The inertia of electromagnetic fields & Maxwell's equations

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This article extends the Electromagnetic (EM) field's Energy-Momentum tensor by adding the missing momentum flux terms. The conservation equations are consequently changed, and as a result the missing electromagnetic terms of inertia forces, centrifugal, Coriolis and shear forces emerge. All the new quantities and the extended quantities are defined as proper covariant tensors. The new definitions of the physical quantities and the extended conservation equations, enable to resolve known problems in classic electrodynamics. Three known problems were chosen to prove the validity of the extended equations and new definitions. The first problem is the 4/3 problem, the second problem is the lack of centrifugal forces in a cylindrical field configuration and the third is the problem of radiation reaction known as the Abraham-Lorentz force. The extended EM motion equations become essential only when very strong EM fields are present or when the inertial terms can not be neglected, otherwise, as in most classical cases in labs, Maxwell's equations are a very good approximation, this is probably why these inertia terms were missed by Maxwell and other classical electrodynamics researchers. The extended motion equations are Non-linear, which means the EM fields can influence themselves or other EM fields, for example a light passing through a strong magnetic field should be bend, this situation is relevant when a light passes a Magnetar's magnetic field. Lastly, the Lagrangian of the extended equation is defined and compared to other two extended nonlinear Lagrangians, the Euler- Heisenberg and Born-Infeld.

I. INTRODUCTION

The Radiation Reaction problem in electrodynamics is at the core of many unsolved problems in classical electrodynamics and QED. The classic problems of accelerated charged particle and others are discussed by Professor Feynman in his Lecture 28-1. According to Feynman these unsolved problems in classic electrodynamics must be solved in the classic picture, since these problems are drudged into quantum mechanics and even into quantum electrodynamics as he said: *"There are difficulties associated with the ideas of Maxwell's theory which are not solved by and not directly associated with quantum mechanics. You may say, Perhaps there's no use worrying about these difficulties. Since the quantum mechanics is going to change the laws of electrodynamics, we should wait to see what difficulties there are after the modification. However, when electromagnetism is joined to quantum mechanics, the difficulties remain "*. In his Lectures Feynman discusses the radiation reaction problem and the 4/3 problem and other unsolved classic problems.

Furthermore, as J.D. Jackson writes in his "Classical Electrodynamics" book (section 17-5), *"... A major problem in the Abraham-Lorentz model is the lack of proper covariance of the electromagnetic self-energy and self-momentum, as manifested by the anomalous factor of 4/3 in the inertia, first found by J. J. Thompson (1881). The root of this difficulty can be traced to the use of the familiar energy and momentum densities,*

$$u = \frac{E}{c^2} = \frac{\mathrm{E}^2 + \mathrm{H}^2}{8\pi c^2}$$

$$\mathbf{g} = \frac{\mathbf{E} \times \mathbf{H}}{\mathbf{4\pi c}}$$

...It is customary to define total electromagnetic energy and momentum as three dimensional volume integrals of these densities at fix time. This is allowable in the discussion of the Poynting theorem for an observer at the rest frame in which the fields are defined (i.e. measured), but is not defensible in general if the total electromagnetic 4-momentum in different inertial frames is to be considered."

Solving these problems is made possible by redefining the EM momentum as an independent covariant tensor uncoupled to Maxwell's stresses tensor unlike Minkowski's definition. Furthermore, the Energy density is newly defined as an independent scalar and not as a component in Minkowski's tensor. Proving the validity of these definitions is quickly shown by solving the 4/3 problem in section V.

While investigating Minkowski's energy momentum tensor, it is apparent that there is a lack of flux terms for the electromagnetic fields. This flux is an intrinsic physical quantity of the momentum and energy of EM fields and will be defined as a covariant quantity. The conservation equations of the new extended EM Energy Momentum tensor has terms that describe the EM inertia quantities like EM centrifugal forces, EM Coriolis forces, etc.

Looking at a situation when a charged particle ,which by definition carries an EM field, is accelerated or moves on a curved path, the inertia forces, such as centrifugal force, will impact the particle. The question is, do these inertial forces impact the EM fields directly, not only by impact on their massive source? According to Einstein EM fields have mass density $\tilde{\rho} = \frac{E}{c^2} = \frac{\mathrm{E}^2+\mathrm{H}^2}{8\pi c^2}$ therefore, we expect the EM fields to change their mass distribution as fluids do.

We can distinguish between two different cases: The first is when a material body source is involved, a charged body, in which case the position, velocity and acceleration can be measured (at least in principle) according to Einstein method of light reflection.

A second example is less familiar, it relates to 'pure' EM momentum density (no massive charged particles are involved, at the volume where the EM fields are investigated). According to Poynting, EM fields contain momentum density vector, $\tilde{p} = \frac{\mathbf{E}\times\mathbf{H}}{4\pi c}$, therefore, EM fields, which have Poynting momentum vector, circulating on a closed path, should experience inertia forces such as centrifugal force. To demonstrate this situation, we take a simple example of a static electric field and perpendicular static magnetic field. The electric field is $\mathrm{E_r}\hat{\mathrm{r}} = \frac{\mathrm{q}}{\mathrm{r}}\hat{\mathrm{r}}$, in the lab it can be created by a long charged wire along the $\hat{z}$ direction. The perpendicular constant magnetic field strength is only on $\hat{z}$, in the lab it can be created by a long solenoid with its axis in the $\hat{z}$ direction, its magnetic field is $\mathrm{H_z}\hat{\mathrm{z}} = \mathrm{Hz}\hat{\mathrm{z}}$, where Hz is constant. The wire is coaxially placed inside the solenoid and we have perpendicular electric and magnetic fields inside the solenoid. These two EM fields hold momentum density on the $\hat{\theta}$ direction, according to Poynting: $\tilde{\mathrm{p}}_\theta = (\frac{\mathrm{E}\times\mathrm{H}}{4\pi c})_\theta = \frac{\mathrm{Hzq}}{4\pi cr}\hat{\theta}$. According to Einstein, the EM fields Energy density also possess mass density $\tilde{\rho} = \frac{Energy\ density}{c^2} = \frac{\mathrm{E}^2+\mathrm{H}^2}{8\pi c^2}$, these EM fields have angular momentum since $\tilde{\mathrm{p}}_\theta \neq 0$, and we expect, just like mechanical fluids, that the EM fields will experience centrifugal force $\frac{\tilde{\mathrm{p}}_\theta^2}{\rho r}$. If these EM fields will not possess any centrifugal force, we will have a contradiction between EM and mechanical inertia -momentum, which makes it difficult to add these physical quantities.

We see that in such cases where inertia of EM fields is present, especially in cosmology and astrophysics, Maxwell's equations cannot give a complete description. In this paper, we extended the motion equations of the EM fields to include the momentum flux and therefore the inertia terms. This extension can answer, in a clear and concise way, the problems mentioned above and many more. The last section of this paper is dedicated to finding a consistent equation of motion for moving charged particle, enabling to describe in one equation the path of the particle and the EM motion including the radiation and its influence in any time on the particle's path.

II. NEW SYMBOLS FOR COMMON TENSOR OPERATIONS

In order to simplify the notations of the repeated mathematical operations, and also to get a better coherent representation of the EM fields interaction, a new symbol of tensorial operation is defined. This operation will help to describe the EM interactions in a more consistent and compact way.

The operation $\dot{\times}$ is defined as the product

For every two tensors A and B

$$\mathrm{A}\dot{\times}\mathrm{B} \equiv \mathrm{A}^{\mu_1,\mu_2\cdots\mu_n}\mathrm{g}_{\mu_n,\nu_1}\mathrm{B}^{\nu_1,\nu_2\cdots\nu_n} \tag{1}$$

We usually operate on tensors with the same rank 2:

$$\mathrm{A}\dot{\times}\mathrm{B} = \mathrm{A}^{\mu\rho}\mathrm{g}_{\rho\lambda}\mathrm{B}^{\lambda\nu}$$

The mathematical properties of the $\dot{\times}$ product are:

I. associative: $\mathrm{A}\dot{\times}(\mathrm{B}+\mathrm{C}) = \mathrm{A}\dot{\times}\mathrm{B} + \mathrm{A}\dot{\times}\mathrm{C}$

II. not commutative: $\mathrm{A}\dot{\times}\mathrm{B} \neq \mathrm{B}\dot{\times}\mathrm{A}$.

Although it is just a private case of tensorial product, we can see the advantages of this product which enables us to directly derive a few EM quantities. We start with the following important example, where F is the field-strength tensor

$$\mathrm{F}^{\mu\nu} = \begin{pmatrix} 0 & -\mathrm{E_x} & -\mathrm{E_y} & -\mathrm{E_z} \\ \mathrm{E_x} & 0 & -\mathrm{H_z} & \mathrm{H_y} \\ \mathrm{E_y} & \mathrm{H_z} & 0 & -\mathrm{H_x} \\ \mathrm{E_z} & -\mathrm{H_y} & \mathrm{H_x} & 0 \end{pmatrix} \tag{2}$$

and $\mathfrak{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\mathrm{F}_{\rho\lambda}$ is its dual field-strength tensor

$$\mathfrak{F}^{\mu\nu} = \begin{pmatrix} 0 & -\mathrm{H_x} & -\mathrm{H_y} & -\mathrm{H_z} \\ \mathrm{H_x} & 0 & \mathrm{E_z} & -\mathrm{E_y} \\ \mathrm{H_y} & -\mathrm{E_z} & 0 & \mathrm{E_x} \\ \mathrm{H_z} & \mathrm{E_y} & -\mathrm{E_x} & 0 \end{pmatrix} \tag{3}$$

using the $\dot{\times}$ product $\frac{1}{4\pi c}(\mathrm{F}\dot{\times}\mathrm{F} + \mathfrak{F}\dot{\times}\mathfrak{F})$ we obtain:

$$\mathrm{T}^{\mu\nu} = \begin{pmatrix} c\tilde{\rho} & \tilde{\mathrm{p}}_x & \tilde{\mathrm{p}}_y & \tilde{\mathrm{p}}_z \\ \tilde{\mathrm{p}}_x & -\frac{\mathrm{E_x}^2+\mathrm{H_x}^2}{4\pi c} + c\tilde{\rho} & -\frac{\mathrm{E_x E_y}+\mathrm{H_x H_y}}{4\pi c} & -\frac{\mathrm{E_x E_z}+\mathrm{H_x H_z}}{4\pi c} \\ \tilde{\mathrm{p}}_y & -\frac{\mathrm{E_x E_y}+\mathrm{H_x H_y}}{4\pi c} & -\frac{\mathrm{E_y}^2+\mathrm{H_y}^2}{4\pi c} + c\tilde{\rho} & -\frac{\mathrm{E_x E_z}+\mathrm{H_x H_z}}{4\pi c} \\ \tilde{\mathrm{p}}_z & -\frac{\mathrm{E_z E_x}+\mathrm{H_z H_x}}{4\pi c} & -\frac{\mathrm{E_z E_y}+\mathrm{H_z H_y}}{4\pi c} & -\frac{\mathrm{E_z}^2+\mathrm{H_z}^2}{4\pi c} + c\tilde{\rho} \end{pmatrix} \tag{4}$$

Where $\tilde{\mathrm{p}} \equiv \frac{\mathrm{E}\times\mathrm{H}}{4\pi c}$ and $\tilde{\rho} \equiv \frac{\mathrm{E}^2+\mathrm{H}^2}{8\pi c^2}$.

This tensor is the known Minikowski's energy-momentum tensor (Jackson 12.10 page 601-605) which is obtained here in a short and simple way by using the $\dot{\times}$ product second order powers of F and $\mathfrak{F}$.

III. A PROPER TENSORIAL DEFINITION OF EM MOMENTUM DENSITY, INDEPENDENT OF MAXWELL'S STRESSES

In 1884 Poynting defined the EM momentum as $\mathbf{p} = \frac{\mathbf{E}\times\mathbf{H}}{4\pi \mathbf{c}}$, it is known as Poynting vector although it is not a vector in R^3 as can be checked by using the mutual tensorial definition of **E** and **H** as $\mathrm{F}^{\mu\nu}$. Later, in 1911 Minkowski defined his covariant energy-momentum tensor and chose the EM energy density to be T^{00} and the EM momentum density to be $\mathrm{T}^{0\mathrm{i}}$, which is still the customary definition (Jackson 17.5). Minkowski related the other components T^{ij} to the known Maxwell stresses, it is important to note that there is no independent (of stresses) covariant definition for the EM momentum and energy density, presently. Therefore, since we use Minkowski's definition of energy and momentum as component of energy-momentum tensor, taking a Lorentz transformation mixes the energy component T^{00} and the momentum components $\mathrm{T}^{0\mathrm{i}}$, with the stress components T^{ij}. This created some problems, probably the famous one is the 4/3 problem.

In this section we are looking for an independent, of stresses, covariant tensor, which will express the EM momentum correctly. Meaning it can replace the previous definition and have added value in solving problems of coupling to stresses etc.

For easier understanding of the algebraic process, we define the following symbols to represent few tensorial products.

For every second degree tensor $A^{\mu\nu}$ and metric $g_{\mu\nu}$, the tensor $\mathrm{A_l} \equiv \mathrm{A_l}{}^{\nu}{}_{\mu} = \mathrm{G}\cdot\mathrm{A} = \mathrm{g}_{\mu\rho}\mathrm{A}^{\rho\nu}$ we call the left mixed tensor

While

For every second degree tensor $A^{\mu\nu}$ and metric $g_{\mu\nu}$, the tensor $\mathrm{A_r} \equiv \mathrm{A_r}{}^{\mu}{}_{\nu} = \mathrm{A}\cdot\mathrm{G} = \mathrm{A}^{\mu\rho}\mathrm{g}_{\rho\nu}$ we call the right mixed tensor.

For general tensor without any special symmetry the mix-tensors, left and right, are not the same tensor.

By using the left and the right mixed tensor we define this tensor:

$$\begin{aligned}\tilde{\mathrm{p}}^{\mu}_{\nu} &\equiv \frac{1}{4\pi c}(g_{\alpha\lambda}\mathfrak{F}^{\lambda\mu}F^{\alpha\rho}g_{\rho\nu} - g_{\alpha\lambda}F^{\lambda\mu}\mathfrak{F}^{\alpha\rho}g_{\rho\nu}) \\ &= \frac{1}{4\pi c}(\mathfrak{F}_{\mathfrak{l}}{}^{\mu}{}_{\lambda}F_{r}{}^{\lambda}{}_{\nu} - F_{l}{}^{\mu}{}_{\lambda}\mathfrak{F}_{\mathfrak{r}}{}^{\lambda}{}_{\nu})\end{aligned} \tag{5}$$

Or in matrix form:

$$\tilde{\mathrm{p}}^{\mu}_{\nu} \equiv \frac{1}{4\pi c}(\mathfrak{F}_{\mathrm{r}}\mathrm{F}_{\mathrm{l}} - \mathrm{F}_{\mathrm{r}}\mathfrak{F}_{\mathrm{l}}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{p}_z & -\tilde{p}_y \\ 0 & -\tilde{p}z & 0 & \tilde{p}_x \\ 0 & \tilde{p}_y & -\tilde{p}_x & 0 \end{pmatrix} \tag{6}$$

Here we regard $\tilde{p}$ only as notation representing the term $\frac{1}{4\pi c}\mathrm{E}\times\mathrm{H}$ and not as a real three vector.

$$\tilde{\mathrm{p}}_{\mu\nu} = \mathrm{g}_{\mu\rho}\tilde{\mathrm{p}}^{\rho}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{p}_z & \tilde{p}_y \\ 0 & \tilde{p}z & 0 & -\tilde{p}_x \\ 0 & -\tilde{p}_y & \tilde{p}_x & 0 \end{pmatrix} \tag{7}$$

We can also define the dual EM momentum tensor as:

$$\tilde{\mathfrak{p}}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\tilde{\mathrm{p}}_{\alpha\beta} = \begin{pmatrix} 0 & \tilde{p}_x & \tilde{p}_y & \tilde{p}_z \\ \tilde{p}_x & 0 & 0 & 0 \\ \tilde{p}_y & 0 & 0 & 0 \\ \tilde{p}_z & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

We choose these tensors Eq.6-8 to represent the independent (of stresses) EM momentum density tensors. We have to be convinced that these tensors are the right choice to represent the EM momentum density, therefore we ask what are the essential characteristics that such EM momentum tensors must have? The first characteristic that the new tensor must have is its proper covariant tensorial structure, this demand is straightforward fulfilled since it was derived from a proper tensorial product. Secondly, since we know that the Poynting vector represents well the EM momentum at the system where the EM fields were measured, we expect the tensor to be build from components of Poynting vector. By looking at the tensors we can see that this characteristic is fulfilled as well. Another important test the new definition must fulfill is the reduction of the tensor when projected on four-vector v^{μ}. This characteristic is evident when we project the tensor Eq.8 on the four-vector v^{μ} in the measuring system, which is by definition always at rest in its own frame of reference, since the rest frame velocity is always: $v^{\mu}_{measure} = (c, 0, 0, 0)$, therefore the projection of the EM momentum Eq.8 on v^{μ} is:

$$\begin{aligned}\mathfrak{p}^{\mu\nu}\mathrm{v}_{\nu} &= (0, \frac{(\mathrm{E}\times\mathrm{H})_{\mathrm{x}}}{4\pi\mathrm{c}}, \frac{(\mathrm{E}\times\mathrm{H})_{\mathrm{y}}}{4\pi\mathrm{c}}, \frac{(\mathrm{E}\times\mathrm{H})_{\mathrm{z}}}{4\pi\mathrm{c}}) \\ &= (0, \tilde{p}_x, \tilde{p}_y, \tilde{p}_z)\end{aligned} \tag{9}$$

The spatial components of this four-vector are exactly the known Poynting three-vector as demanded.

Applying the Lorentz transformation on Eq.7 or Eq.8 will give a tensor which contains only the Poynting components, free of stresses.

Taking the second power of the EM momentum tensor under the product $\dot{\times}$:

$$\tilde{\mathrm{P}}^2 \equiv \tilde{\mathrm{P}}\dot{\times}\tilde{\mathrm{P}} = \begin{pmatrix} \tilde{p}^2 & 0 & 0 & 0 \\ 0 & -\tilde{p}_x\tilde{p}_x & -\tilde{p}_x\tilde{p}_y & -\tilde{p}_x\tilde{p}_z \\ 0 & -\tilde{p}_y\tilde{p}_x & -\tilde{p}_y\tilde{p}_y & -\tilde{p}_x\tilde{p}_z \\ 0 & -\tilde{p}_z\tilde{p}_x & -\tilde{p}_z\tilde{p}_y & -\tilde{p}_z\tilde{p}_z \end{pmatrix} \tag{10}$$

Where $\tilde{p}^2 \equiv \tilde{p}_x^2 + \tilde{p}_y^2 + \tilde{p}_z^2 = \frac{1}{16\pi^2 c^2}(\mathrm{E}^2\mathrm{H}^2 - (\mathrm{E}\cdot\mathrm{H})^2)$.

Since the trace of a tensor is always a scalar $\frac{1}{2}Trace\left(\tilde{\mathrm{P}}^2\right) \Rightarrow \frac{1}{2}\mathrm{g}_{\mu\mu}\tilde{\mathrm{P}}^{\mu\mu} = \tilde{p}^2$. Therefore we define the EM momentum norm as the square root $\|\mathbf{p}\| = \sqrt{\frac{1}{2}\mathrm{g}_{\mu\mu}\tilde{\mathrm{P}}^{\mu\mu}}$. In the measuring system the tensorial momentum density norm looks the same as the known covariant Poynting vector norm, but the difference here is that the momentum tensor and its norm are properly defined.

This newly defined EM momentum tensor has an important algebraic characteristic under the product $\dot{\times}$ powers: It behaves as an algebraic group of two components. The third power of the momentum tensor is: $\mathrm{P}^2\dot{\times}\mathrm{P} = \frac{1}{2}\mathrm{Trace}\left(\mathrm{P}^2\right)\mathrm{P}^{\mu\nu}$, which is again the original momentum tensor $\mathrm{P}^{\mu\nu}$ multiplied by the scalar: $\frac{1}{2}Trace\left(\mathrm{P}^2\right)$. The fourth power of the momentum tensor will give again the second power P^2 multiplied by the same scalar twice, and so on.

This makes the momentum tensor under the product $\dot{\times}$ an algebraic group with two elements P and P^2. The operation $\frac{1}{\frac{1}{2}Trace\mathrm{P}^2}\dot{\times}$, will help us later to define the general form of the EM momentum flux tensor.

The new momentum density tensors Eq.6-8 do not include energy density component, while Minikowski's energy-momentum tensor Eq.2 does have an energy density component T^{00} . Therefore, we need to define an EM energy density, which is not a component of Minikowski's tensor and should fulfill these five restrictions:

1. It should only be a combination of $\mathrm{F}^{\mu\nu}$ and $\mathfrak{F}^{\mu\nu}$.

2. There will be no use of any new units-constants.

3. It should always be positive.

4. In the system where E and H are measured, it should become the known expression for EM energy density: $\tilde{\rho} = \frac{1}{8\pi}\left(\mathrm{E}^2{+}\mathrm{H}^2\right)$.

5. It has to be a scalar quantity for the following reasons: The 3-D surface element is a proper four-vector, its form in the rest frame is: $d\Sigma^\mu = (d^3x, 0, 0, 0)$. Furthermore, the energy density $\tilde{\rho}$ multiplied by the deferential volume d^3x by definition must be the mass in this deferential volume dm. The only way to comply with Einstein four-momentum definition is to take the deferential mass $cdm = c\tilde{\rho}d^3x$ to be the time component of the four-momentum :

$$dp_\mu = \tilde{\rho}\Sigma^\mu = (c\tilde{\rho}d^3x, 0, 0, 0) = (cdm, 0, 0, 0) \tag{11}$$

In order for Eq.11 to be a proper four-vector we must multiply $d\Sigma^\mu$ by a scalar function $\tilde{\rho}$. In all cases where $\tilde{\rho}$ is not a proper scalar, the $\tilde{\rho}d\Sigma^\mu$ will never be a proper four vector.

From Eq.11 we can derive the total mass in the volume V, which is also a proper four-vector:

$$\int_V dp_\mu = (\int_V c\tilde{\rho}d^3x, 0, 0, 0) = (cm, 0, 0, 0) \tag{12}$$

In the moving frame, on x direction it becomes, the known expression: $p^\mu = (m\gamma c, m\gamma v, 0, 0)$.

This leaves us with only one suitable expression for the EM energy density which fulfills all these requirements:

$$\widetilde{\rho} = \frac{1}{8\pi}\sqrt{\left(\frac{1}{2}\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}\right)^2 + \left(\frac{1}{2}\mathrm{F}_{\mu\nu}\mathfrak{F}^{\mu\nu}\right)^2 + 2\mathrm{P}_{\mu\nu}\mathrm{P}^{\mu\nu}} \tag{13}$$

Checking the restrictions: All the terms are a combination of the field strength tensor $\mathrm{F}^{\mu\nu}$ and its dual $\mathfrak{F}^{\mu\nu}$. The first and second terms are positive by being square of real function, the third term in details is $2(\mathrm{p}_\mathrm{x}^2 + \mathrm{p}_\mathrm{y}^2 + \mathrm{p}_\mathrm{z}^2) \geq 0$. No new constants are used, therefore three requirements are fulfilled. The fourth requirement refers to the form in the system where the EM fields where measured:

$$\widetilde{\rho} = \frac{1}{8\pi}\sqrt{(\mathrm{E}^2 - \mathrm{H}^2)^2 + (2\mathrm{E}\cdot\mathrm{H})^2 + (2\mathrm{E}\times\mathrm{H})^2} \tag{14}$$

Since $(\mathrm{E}\times\mathrm{H})^2 = \mathrm{E}^2\mathrm{H}^2 - (\mathrm{E}\cdot\mathrm{H})^2$ we obtain: $\tilde{\rho} = \frac{1}{8\pi}(\mathrm{E}^2 + \mathrm{H}^2)$ as expected. For the last requirement the three terms under the square root of Eq.13 are each a reduction of rank two covariant and contravariant proper tensors, which gives by definition three proper scalars, making their sum a scalar. Therefore the last demand is also fulfilled.

This new energy density scalar is important for resolving difficulties in electrodynamics especially in general relativity electrodynamics.

Now we can define another important quantity, the covariant EM velocity tensor, as:

$$\tilde{\mathrm{u}}^{\mu\nu} \equiv \frac{\tilde{\mathfrak{p}}^{\mu\nu}}{\tilde{\rho}} = \begin{pmatrix} 0 & \tilde{\mathrm{u}}_\mathrm{x} & \tilde{\mathrm{u}}_y & \tilde{\mathrm{u}}_z \\ \tilde{\mathrm{u}}_\mathrm{x} & 0 & 0 & 0 \\ \tilde{\mathrm{u}}_y & 0 & 0 & 0 \\ \tilde{\mathrm{u}}_z & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

This velocity describes an intrinsic property of the EM fields, expressing the rate in which EM momentum is transferred from one point to another inside the volume where the EM field exists. Usually the EM velocity cannot be measured directly as mechanical velocity of material body which can reflect light as Einstein's measuring technique requires. Therefore, in most cases it doesn't have a simple connection to the mechanical velocity of Newton or Einstein, an exceptional case is a free plane wave.

To observe more closely the connection we take a plane wave solution which moves on the $\hat{z}$ direction, the electric field is on the $\hat{x}$ direction, $\mathbf{E}(z,t) = \mathrm{E_0cos(kz - \omega t)}\hat{\mathrm{x}}$ and $\mathbf{H}(z,t) = \mathrm{H_0cos(kz - \omega t)}\hat{\mathrm{y}}$. Inserting in the electromagnetic tensor F gives:

$$\begin{pmatrix} 0 & \mathrm{E_0cos(kz - \omega t)} & 0 & 0 \\ -\mathrm{E_0cos(kz - \omega t)} & 0 & 0 & -\mathrm{H_0cos(kz - \omega t)} \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{H_0cos(kz - \omega t)} & 0 & 0 \end{pmatrix} \tag{16}$$

Now we have to calculating $\tilde{\mathfrak{p}}^{\mu\nu}$ in Eq.8 and by using the fact that in vacuum $\|\mathbf{E}\| = \|\mathbf{H}\|$ we get

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{E_0^2}{4\pi c}cos^2(kz - \omega t) & 0 \\ 0 & -\frac{E_0^2}{4\pi c}cos^2(kz - \omega t) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

We calculate $\tilde{\rho}$ by installing in Eq.13 and we get $\tilde{\rho} = \frac{1}{4\pi c^2}\mathrm{E_0^2cos^2(kz - \omega t)}$. The EM velocity is $\tilde{\mathbf{u}}^{\mu\nu} = \tilde{\mathbf{p}}^{\mu\nu}/\tilde{\rho}$ which is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

or in non covariant form $\tilde{\mathbf{u}} = (\mathbf{0}, \mathbf{0}, \mathbf{c})$. This example is an exception but is a very important one, since it correlates Einstein's velocity measuring method, using EM waves to the EM velocity. A free EM wave is "pure" EM phenomena, where "pure" means there is no material bodies involved at least while the wave is propagating. On the other hand the EM waves are created by manipulating a material body and also are measured by a material device which has mechanical properties.

Remark: *In present literature, for example [12], only two fundamental invariants of* $\mathrm{F}^{\mu\nu}$ *and* $\mathfrak{F}^{\mu\nu}$ *exist, where 'fundamental' means all other invariants can be expressed as algebraic combination of these two:* $\mathrm{F}^{\mu\nu}\mathrm{F}_{\mu\nu} = \mathfrak{F}^{\mu\nu}\mathfrak{F}_{\mu\nu} = -2\left(\mathrm{E}^2 - \mathrm{H}^2\right)$ *and* $\mathrm{F}_{\mu\nu}\mathfrak{F}^{\mu\nu} = -4E\cdot B$ *The new* *invariant* $\widetilde{\mathrm{P}}_{\mu\nu}\widetilde{\mathrm{P}}^{\mu\nu}$ *in the definition of* $\tilde{\rho}$*, is obtained by using the* $\dot{\times}$ *product fourth power of* F *and* $\mathfrak{F}$*, it cannot be obtained by algebraic operations of second power of* F *and* $\mathfrak{F}$*. Therefore it remained unknown until these definitions.*

A. EM fields attached to a material body

When EM fields are attached to a charged moving body which has a mechanical velocity $\mathbf{v}$, we can use the EM energy density scalar and the mechanical four-velocity of the body, to define a proper four-vector momentum for the attach EM fields:

$$p_\mu \equiv \tilde{\rho} v_\mu = (\tilde{\rho}\gamma, \tilde{\rho}\gamma\mathbf{v}) \tag{19}$$

We can also define an energy-momentum flux tensor for the EM fields moving with the mechanical body as:

$$P^{\mu\nu} \equiv p^\mu p^\nu / \tilde{\rho} = \tilde{\rho} v^\mu v^\nu \tag{20}$$

The projection of the 'pure' EM momentum tensor density Eq.8 on the four-velocity $v^\mu_{rest} = (c, 0, 0, 0)$, gives:

$$\tilde{p}^\mu = \tilde{\mathrm{P}}^\mu_{\ \nu} v^\nu_{(rest)} = (0, \tilde{\mathrm{p}}_{\mathrm{x}}, \tilde{\mathrm{p}}_{\mathrm{y}}, \tilde{\mathrm{p}}_{\mathrm{z}})$$

The spatial components have the same form as Poynting Vector in the system where the fields E and H where measured.

In certain situations when both a 'pure' EM field (free field) and an attached EM field exist in the same volume, their mutual four-momentum vector is their sum $p^\mu_T \equiv p^\mu + \tilde{p}^\mu$.

Therefore, representing the total four momentum of a charged particle and its fields with external EM fields represented by $\tilde{p}^\mu_{Free}$ is:

$$p^\mu_{Total} = p^\mu{}_{Mechanical} + p^\mu{}_{Attached} + \tilde{p}^\mu_{Free} \tag{21}$$

B. Application to relativistic fluids

Although it may seem indirectly related to the main subject of this paper, we chose a problem in the mass density of relativistic fluids which is resolved, by these new definitions, at least in some cases.

In most literature the mass density of fluid is defined as $\rho = \frac{\mathrm{dm}}{d^3x}$ [20] where m is the mass and $d^3x = d\mathrm{v}$ is the volume but is not a scalar as can be checked: $\rho^{'} = \frac{\mathrm{d^3m}^{'}}{d^3x'} = \gamma^2\rho$, in fact this term transforms as the T^{00} component, where $\mathrm{T}^{\mu\nu}$ is the fluid energy-momentum tensor.

In fluid dynamics the flux tensor $\rho v^\mu v^\nu$ is not a proper tensor, although $v^\mu v^\nu$ is a proper tensor, the mechanical density $\rho = \frac{dm}{dV}$ is not a scalar, therefore their multiplication does not transform as a proper tensor. Einstein, in his famous 'General Relativity' paper[3], when introducing this energy-momentum tensor noticed this problem and writes: *'Let* ρ *and* p *be two scalars, the former of which we call "density", the latter the "pressure" of a fluid and let an equation subsist between them'.*

Einstein never mentioned $\rho = \frac{\mathrm{dm}}{d^3x}$ as the definition for density since he knew it's not a proper scalar. The new definition of EM mass density $\tilde{\rho}$ Eq.13 is the only proper scalar that solves this problem at least in the case where only EM fields are the source of mass density. If we take the mass of particles as non-electromagnetic in origin, we cannot define the mass density as a scalar. But, if we could have assumed that all material (elementary particles) are in essence electromagnetic in nature, as Feynman said in Lecture 28-3 *"... there is the thrilling possibility that the mechanical piece is not there at all—that the mass is all electromagnetic."* then the scalar EM mass density solves this problem.

IV. USING THE NEW DEFINITIONS TO RESOLVE THE 4/3 PROBLEM

The 4/3 problem deals with charged particle with constant mechanical velocity v. In its rest frame the particle has a Coulomb field $\mathbf{E} = \frac{\mathbf{q}}{\mathbf{r^2}}\hat{\mathbf{r}}$, the known energy-momentum tensor $T^{\mu\nu}$ for this case, in the particle rest frame, is:

$$\frac{1}{4\pi}\begin{pmatrix} \frac{1}{2}E^2 & 0 & 0 & 0 \\ 0 & -E_x{}^2 + \frac{1}{2}E^2 & -E_xE_y & -E_xE_z \\ 0 & -E_yE_x & -E_y{}^2 + \frac{1}{2}E^2 & -E_yE_z \\ 0 & -E_zE_x & -E_zE_y & -E_z{}^2 + \frac{1}{2}E^2 \end{pmatrix} \tag{22}$$

The Lorentz transformation on the x̂ direction gives

$$T'^{00} = \gamma^2(T^{00} + \beta^2 T^{11}) \quad T'^{01} = \beta\gamma^2(T^{00} + T^{11}) \tag{23}$$

To understand what the problem is, we follow the treatment in the book of Becker [5], which was originally written by Max Abraham. Thompson, Abraham, Lorentz and Poincare were the main pioneers that tried to solve the 4/3 paradox. Abraham in his book, used $d^3x = \gamma d^3x'$ or $d^3x' = \gamma^{-1}d^3x$ as the volume element in the rest frame to integrated overall space. He used the first equation of Eq.23, which he presumed represents the moving energy density in the rest frame, therefore the total mass of the field is: $m^{'} \equiv \frac{1}{c^2}\int T'^{00} d^3x'$ or:

$$m^{'} = \frac{1}{c^2\sqrt{1-\beta^2}}\int (T^{00} + \beta^2 T^{11}) d^3x$$

Using the same approach, $\mathrm{P}'^1 \equiv \frac{1}{c^2}\int \quad \mathrm{T}'^{01}\mathrm{d^3x'}$, or in detail:

$$\mathrm{P}'^1 = \frac{\mathrm{v}}{\mathrm{c}^2\sqrt{1-\beta^2}}\int (\mathrm{T}^{00} + \mathrm{T}^{11})\mathrm{d^3x}$$

which Abraham presumed represents the overall EM momentum, on the $\hat{x}$ direction. The integral over T^{11} includes integration over the three components of $\mathbf{E}$, since the electric field is isotropic: $\int(E_x^2)d^3x = \int(E_y^2)d^3x = \int(E_z^2)d^3x = \frac{1}{3}U$. Therefore installing this in the integral above gives:

$$\mathbf{P'^1} = \frac{\mathbf{4U_0}}{\mathbf{3c^2}}\frac{\mathbf{v}}{\sqrt{\mathbf{1}-\beta^\mathbf{2}}} = \frac{\mathbf{4}}{\mathbf{3}}\mathbf{m_{EM-fields}}\frac{\mathbf{v}}{\sqrt{\mathbf{1}-\beta^\mathbf{2}}} \quad (24)$$

There is an added mass of $\frac{1}{3}m_{EM-fields}$ which is not expect from Einstein's definition for mass energy relation $m = \mathrm{U/c^2}$, and also as measured by experiment.

Resolving the 4/3 problem:

At the base of this problem/paradox is a wrong assumption that the EM energy-momentum tensor $\mathrm{T}^{\mu\nu}$ components $\mathrm{T}^{0\mu}$ represents the energy momentum density of EM fields.

Resolving is straightforward, we simply have to use correct covariant definition of the EM energy and momentum of a charged particle defined above by Eq.19. Inserting in Eq.19 the case of charged particle in a rest frame $\mathbf{v} = \mathbf{0}$, Eq.19 becomes:

$$\tilde{\mathrm{P}}_\mu = (\mathrm{c}\tilde{\rho}, 0, 0, 0) \quad (25)$$

As defined in the rest frame, where only $\mathbf{E}$ exists the EM energy/mass density Eq.14 becomes $\tilde{\rho} = \frac{\mathbf{E^2}}{8\pi c^2}$. Since it is a proper scalar it remains the same in all moving frames therefore, when the particle is moving on the $\hat{x}$ axis $(\gamma, v, 0, 0)$ Eq.19, after taking Lorentz transformation, becomes:

$$\tilde{\mathrm{P}}'_\mu = (\gamma\mathrm{c}\tilde{\rho}, \gamma\mathrm{v}\tilde{\rho}, 0, 0) \quad (26)$$

To find the overall EM energy-momentum four-vector, we need to integrate in the measuring/lab system the moving body EM energy-momentum four-vector Eq.19

$$\begin{aligned}\int[\tilde{\mathrm{P}}']_\mu \mathrm{d^3x} &= \left(\int \gamma\mathrm{c}\tilde{\rho}\mathrm{d^3x}, \int \gamma\mathrm{v}\tilde{\rho}\mathrm{d^3x}, 0, 0\right) \\ &= (\gamma cm, \gamma mv, 0, 0)\end{aligned} \quad (27)$$

With this proper four-vector definition the excess 1/3 mass is gone and we get exactly the same form as the mechanical four-momentum as Einstein's relativity demands.

V. THE EXTENDED MINKOWSKI'S EM ENERGY-MOMENTUM TENSOR AND ITS CONSERVATION EQUATIONS

We start with motivation arguments for extending the EM energy-momentum tensor. The first argument comes from comparing the structure of the EM energy-momentum tensor to the relativistic-fluid energy-momentum tensor. The Minkowski EM tensor Eq.4 can be written as:

$$\begin{pmatrix} \mathrm{c}\tilde{\rho} & \tilde{\mathrm{p}}_\mathrm{x} & \tilde{\mathrm{p}}_\mathrm{y} & \tilde{\mathrm{p}}_\mathrm{z} \\ \tilde{\mathrm{p}}_\mathrm{x} & \mathrm{s}^{11} & \mathrm{s}^{12} & \mathrm{s}^{13} \\ \tilde{\mathrm{p}}_\mathrm{y} & \mathrm{s}^{21} & \mathrm{s}^{22} & \mathrm{s}^{23} \\ \tilde{\mathrm{p}}_\mathrm{z} & \mathrm{s}^{31} & \mathrm{s}^{32} & \mathrm{s}^{33} \end{pmatrix} \quad (28)$$

Where $\tilde{\rho} = \frac{\mathrm{E^2+H^2}}{8\pi c^2}$ and $\tilde{\mathrm{p}} = \frac{\mathrm{E\times H}}{4\pi\mathrm{c}}$ the mass and momentum density of the EM field, and s^{ij} are the Maxwell's stresses divided by $\frac{1}{4\pi c}$:

$$\mathrm{S^{ij}} = -\frac{1}{4\pi c}\left(\mathrm{E^iE^j} + \mathrm{H^iH^j}\right) + \left(\frac{\mathrm{E^2+H^2}}{8\pi c}\right)\delta^{\mathrm{ij}} \quad (29)$$

For comparison, the fluid-mechanics energy-momentum tensor divided by $\gamma^2 c$ becomes :

$$T_{fluids} = \begin{pmatrix} c\rho & p_x & p_y & p_z \\ p_x & \sigma^{11} + \frac{\rho}{c}u_xu_x & \sigma^{12} + \frac{\rho}{c}u_xu_y & \sigma^{13} + \frac{\rho}{c}u_xu_z \\ p_y & \sigma^{21} + \frac{\rho}{c}u_yu_x & \sigma^{22} + \frac{\rho}{c}u_yu_y & \sigma^{23} + \frac{\rho}{c}u_yu_z \\ p_z & \sigma^{31} + \frac{\rho}{c}u_zu_x & \sigma^{32} + \frac{\rho}{c}u_zu_y & \sigma^{33} + \frac{\rho}{c}u_zu_z \end{pmatrix} \quad (30)$$

In this tensor mechanical quantities are written without tilled letters.

The fluid energy-momentum conservation equations are

$$\partial_\mu T^{\mu\nu}_{fluids} = 0 \quad (31)$$

These are the relativistic motion equations or Einstein-Euler relativistic equations. The fluid momentum is: $\mathrm{p_i(t,\mathbf{x})} = \rho\frac{\partial\mathbf{x_i}}{\partial\mathbf{s}} = \rho\mathrm{u_i(t,\mathbf{x})}$. The stresses are $\sigma^{\mathrm{ij}} = (\mathrm{g^{ij}P + f^{ij}} + \nu^{\mathrm{ij}})/\mathrm{c}\gamma^2$, where P is the pressure, $\mathrm{f^{ij}}$ stand for other kind of stresses and ν^{ij} stand for friction stresses i.e. viscosity.

The four-divergence gives the conservation equations:

$$\dot{\rho} + \nabla\cdot(\rho\mathbf{u}) = 0 \quad (32)$$

and

$$(\dot{\rho u^j}) + \partial_i(\rho u^i u^j) + \partial_i\sigma^{ij} = 0 \quad (33)$$

The divergence of Minkowski EM momentum-energy tensor $\partial_\mu \mathrm{T}^{\mu\mathrm{i}} = 0$, are the EM energy-momentum conservation equations:

$$\dot{\tilde{\rho}} + \nabla\cdot(\tilde{\rho}\tilde{\mathbf{u}}) = 0 \quad (34)$$

$$(\dot{\tilde{\rho}\tilde{u}^j}) + \partial_i s^{ij} = 0 \quad (35)$$

Where:

$\partial_i s^{ij} = -\frac{1}{4\pi}[\mathbf{E}\nabla\cdot\mathbf{E} + \mathbf{H}\nabla\cdot\mathbf{H} + \mathbf{E}\times(\nabla\times\mathbf{E}) + \mathbf{H}\times(\nabla\times\mathbf{H})]^\mathbf{j}$

Comparing the mechanical motion equations Eq.33 to the EM motion equations Eq.35, we see missing terms: $\partial_i(\rho u^i u^j)$. These terms are the divergence of the fluid's momentum flux $\rho u^i u^j$, it is evident that there is no representation of the momentum flux of EM fields in the EM conservation equations.

The terms $\partial_i(\rho u^i u^j)$ in the mechanical case, are the centrifugal forces and Coriolis forces as seen in cylindrical coordinates. Inserting the continuity equation Eq.31 into Eq.33 gives Euler's equations:

$$\rho \dot{u}^j + \partial_i(\rho u^i u^j) = \rho \dot{u}^j + \rho(u^i \partial_i u^j)$$

In cylindrical coordinates Euler's equations of ideal fluid, become:

$$\rho(u^i \partial_i u^1) = \rho u_r \frac{\partial u_r}{\partial r} + \rho \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \rho u_z \frac{\partial u_z}{\partial z} - \rho \frac{{u_\theta}^2}{r} \quad (36)$$

$$\rho(u^i \partial_i u^2) = \rho u_r \frac{\partial u_\theta}{\partial r} + \rho \frac{u_\theta}{r}\frac{\partial\ u_\theta}{\partial \theta} + \rho u_z \frac{\partial u_\theta}{\partial z} + \rho \frac{u_r u_\theta}{r} \quad (37)$$

$$\rho(u^i \partial_i u^3) = \ \rho u_r \frac{\partial u_z}{\partial r} + \rho \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + \rho u_z \frac{\partial u_z}{\partial z} \quad (38)$$

The term $\rho\frac{{u_\theta}^2}{r}$ in Eq.37 is the fluid centrifugal force and the term $\rho\frac{u_r u_\theta}{r}$ in Eq.38 is the fluid Coriolis force. The absence of these inertial force terms in Eq.35, means that EM fields do not possess inertial forces according to Maxwell's equations. We might think this is the nature of EM fields, but we know EM fields contain energy density using: $m = U/c^2$, this imposes on the EM fields to have mass density. Anything that has mass by definition contains inertia. Still one can say although EM fields contain mass their centrifugal force is somehow canceled out as Maxwell's equations suggest. It will be very strange and even will cause a contradiction, for example when EM fields are rotating while attached to their rotating source. The rotating body experiences centrifugal force, how then, the attached EM fields that have mass and rotate with it do not experience centrifugal forces?

Another fundamental example which demonstrates the need for flux terms, as presented in the introduction, is a long cylindrical electrode, homogeneously charged, installed concentrically within a long solenoid. The electric field inside the cylinder, using Maxwell equation or Gauss law, is $\mathbf{E} = \frac{\lambda}{\mathbf{r}}\hat{\mathbf{r}}$ where λ is charge per unit length. The magnetic field inside the solenoid using Maxwell equation or Ampere's Law is $\mathbf{H} = \mathrm{H_0}\hat{\mathrm{z}}$, where $\mathrm{H_0} = \mathrm{Const}$ proportional to the solenoid current per unit length. The fields inside the solenoid have EM momentum density according to Poynting definition, which is proper to be used in the measuring system, $\tilde{\mathbf{p}} = \frac{\lambda \mathrm{H_0}}{\mathbf{4\pi cr}}\hat{\theta}$. Since EM momentum is part of the overall momentum of any system, the EM momentum can be added and also transformed to mechanical momentum and vice-versa. This means the fields momentum in $\hat{\theta}$ must possess angular momentum density $\tilde{\mathbf{L}} = \mathrm{r} \times \tilde{\mathbf{p}} = \mathrm{r}\tilde{\mathrm{p}}_\theta$ as rotating fluids would have in the corresponding situation. These EM fields must also possess centrifugal force density, since $\tilde{\mathbf{f}}_\mathbf{c} = \frac{\tilde{\mathrm{p}}_\theta^2}{\tilde{\rho}\mathbf{r}} = \tilde{\rho}\frac{\tilde{\mathrm{u}}_\theta^2}{\mathbf{r}}$, where all quantities are EM.

From the reasons stated above we conclude, the EM energy momentum tensor should include flux terms. For this we have the following requirements: The extended energy-momentum tensor has to be a proper covariant tensor, therefore the flux terms should be an addition of a proper covariant flux tensor. The flux tensor should be composed only from the EM tensors F or $\mathfrak{F}$ and the metric tensor, meaning no mechanical quantities should be involved since no materials are present only EM fields. Another requirement is that the flux tensor should be symmetric since the Minkowski's energy-momentum tensor is. And the last requirement comes from the structure of the mechanical flux tensor, since all other terms in both tensors have similar structure, the flux tensor should be similar in structure to the fluid case.

The tensor which answers all these demands is:

$$\tilde{\mathrm{M}}^{\mu\nu} \equiv \frac{1}{\mathrm{c}\tilde{\rho}}\tilde{\mathrm{P}}\dot{\times}\tilde{\mathrm{P}} = \frac{1}{c}\begin{pmatrix} -\tilde{\rho}\tilde{\mathrm{u}}^2 & 0 & 0 & 0 \\ 0 & \tilde{\rho}\tilde{\mathrm{u}}_\mathrm{x}\tilde{\mathrm{u}}_\mathrm{x} & \tilde{\rho}\tilde{\mathrm{u}}_\mathrm{x}\tilde{\mathrm{u}}_\mathrm{y} & \tilde{\rho}\tilde{\mathrm{u}}_\mathrm{x}\tilde{\mathrm{u}}_\mathrm{z} \\ 0 & \tilde{\rho}\tilde{\mathrm{u}}_\mathrm{y}\tilde{\mathrm{u}}_\mathrm{x} & \tilde{\rho}\tilde{\mathrm{u}}_\mathrm{y}\tilde{\mathrm{u}}_\mathrm{y} & \tilde{\rho}\tilde{\mathrm{u}}_\mathrm{y}\tilde{\mathrm{u}}_\mathrm{z} \\ 0 & \tilde{\rho}\tilde{\mathrm{u}}_\mathrm{z}\tilde{\mathrm{u}}_\mathrm{x} & \tilde{\rho}\tilde{\mathrm{u}}_\mathrm{z}\tilde{\mathrm{u}}_\mathrm{y} & \tilde{\rho}\tilde{\mathrm{u}}_\mathrm{z}\tilde{\mathrm{u}}_\mathrm{z} \end{pmatrix} \quad (39)$$

This tensor satisfied all the above demands for EM momentum flux tensor: It is symmetric, constructed only from F and $\mathfrak{F}$, it has the right units, it has similar structure as the relativistic mechanics-fluid flux momentum tensor, we just have to replace $\tilde{\rho} \to \rho$ and $\tilde{\mathbf{u}}_\mathrm{i} = (\mathbf{u}_{EM})_\mathrm{i} \to (\mathbf{u}_{Mechanics})_\mathrm{i}$.

Although Eq.39 satisfies our demands, we can still ask, is it the most extended flux tensor and is it the only one? In principle we can compose other tensors that have the right units and symmetry and do represent momentum flux tensor for example: $\mathrm{M}^{\mu\nu} \equiv \tilde{\rho}\mathrm{u}^\mu \mathrm{u}^\mu$. Although this is a proper covariant tensor describing an EM momentum flux, it can not be part of the extended EM tensor since, it contains a mechanical velocity u^μ, the demand is that no mechanical quantities are involved in this extended EM tensor.

It still does not prove that higher powers of $\tilde{\mathrm{P}}^{\mu\nu}$ cannot be part of the extend flux tensor Eq.39. To prove it is not possible, we use the algebraic group characteristic of the EM momentum tensor (see section III). If we take higher powers of the EM momentum tensor, the multiplied tensor returns to one of the two terms in the group, one is not the regular momentum which is not suitable and the other is momentum flux that exist.

All the arguments above suggest that the most natural possibility for extending the EM energy-momentum tensor is the EM flux tensor Eq.39. The extension is done by simply adding it to Minkowski's energy-momentum tensor Eq.4:

$$\tilde{\mathrm{T}}^{\mu\nu} = \mathrm{T}^{\mu\nu} + \tilde{\mathrm{M}}^{\mu\nu} \quad (40)$$

The extended 'pure' EM energy-momentum tensor (divided by c) can be written as powers of $\mathrm{F}^{\mu\nu}$ and $\mathfrak{F}^{\mu\nu}$

simply as:

$$\tilde{\mathrm{T}} = \frac{1}{8\pi c}\left(\mathrm{F}\dot{\times}\mathrm{F} + \mathfrak{F}\dot{\times}\mathfrak{F}\right) + \frac{1}{c\tilde{\rho}}\tilde{\mathrm{P}}\dot{\times}\tilde{\mathrm{P}} \tag{41}$$

For convenience we define $\tilde{\gamma} = \frac{1}{\sqrt{1-\frac{\tilde{u}^2}{c^2}}}$, notice we used the EM velocity $\tilde{u}$ not the mechanical velocity as we use in γ without tilde. Therefore Eq.42 details the extended EM energy-momentum tensor:

$$\tilde{\mathrm{T}}^{\mu\nu} = \begin{pmatrix} \mathrm{c}\tilde{\rho}\tilde{\gamma}^{-2} & \tilde{\mathrm{p}}_\mathrm{x} & \tilde{\mathrm{p}}_\mathrm{y} & \tilde{\mathrm{p}}_\mathrm{z} \\ \tilde{\mathrm{p}}_\mathrm{x} & \mathrm{s}^{11} + \frac{\tilde{\rho}}{\mathrm{c}}\tilde{\mathrm{u}}_\mathrm{x}\tilde{\mathrm{u}}_\mathrm{x} & \mathrm{s}^{12} + \frac{\tilde{\rho}}{\mathrm{c}}\tilde{\mathrm{u}}_\mathrm{x}\tilde{\mathrm{u}}_\mathrm{y} & \mathrm{s}^{13} + \frac{\tilde{\rho}}{\mathrm{c}}\tilde{\mathrm{u}}_\mathrm{x}\tilde{\mathrm{u}}_\mathrm{z} \\ \tilde{\mathrm{p}}_\mathrm{y} & \mathrm{s}^{21} + \frac{\tilde{\rho}}{\mathrm{c}}\tilde{\mathrm{u}}_\mathrm{y}\tilde{\mathrm{u}}_\mathrm{x} & \mathrm{s}^{22} + \frac{\tilde{\rho}}{\mathrm{c}}\tilde{\mathrm{u}}_\mathrm{y}\tilde{\mathrm{u}}_\mathrm{y} & \mathrm{s}^{23} + \frac{\tilde{\rho}}{\mathrm{c}}\tilde{\mathrm{u}}_\mathrm{y}\tilde{\mathrm{u}}_\mathrm{z} \\ \tilde{\mathrm{p}}_\mathrm{z} & \mathrm{s}^{31} + \frac{\tilde{\rho}}{\mathrm{c}}\tilde{\mathrm{u}}_\mathrm{z}\tilde{\mathrm{u}}_\mathrm{x} & \mathrm{s}^{32} + \frac{\tilde{\rho}}{\mathrm{c}}\tilde{\mathrm{u}}_\mathrm{z}\tilde{\mathrm{u}}_\mathrm{y} & \mathrm{s}^{33} + \frac{\tilde{\rho}}{\mathrm{c}}\tilde{\mathrm{u}}_\mathrm{z}\tilde{\mathrm{u}}_\mathrm{z} \end{pmatrix} \tag{42}$$

We notice that no new physical constants are needed, indicating that this extension of Maxwell's equations in vacuum contains only EM fields, no need for the electron's charge or its mass or $\hbar$ as other extensions of Maxwell's equations use, like, Euler- Heisenberg Lagrangian or Born-Infeld nonlinear Electrodynamics. The trace of Eq.39 by definition should be a scalar, we get $c\tilde{\rho}$ as expected.

Now that the extended tensor is defined, what should be the generalized energy-momentum conservation equations? The natural choice is the four-divergence of the new tensor. The first reason is that for $\tilde{u}^\mu = 0$, we should get the Minikowski's conservation equations.

Another reason is that the four-divergence gives the EM energy-momentum conservation equations which mean, that no energy or momentum escape from the volume, where the fields are, which has a time-like surface S. This demand is written as:

$$\int\limits_S \tilde{\mathrm{T}}^{\mu\nu} d^3\sigma_\mu \Rightarrow \int\limits_S \left(\mathrm{T}^{\mu\nu} + \tilde{\mathrm{M}}^{\mu\nu}\right) d^3\sigma_\mu = 0 \tag{43}$$

Applying the divergence theorem we obtain:

$$\int\limits_V \partial_\mu \tilde{\mathrm{T}}^{\mu\nu} d^4\mathrm{x} = 0 \tag{44}$$

A sufficient condition that these equations are fulfilled for such a volume is:

$$\partial_\mu \tilde{\mathrm{T}}^{\mu\nu} = 0 \tag{45}$$

This equation is the generalized energy-momentum conservation equation.

We can verify that the additional terms, the momentum flux terms, are negligible when $\frac{\tilde{\mathrm{u}}_\mathrm{i}}{c} \ll 1$:

$$\tilde{\mathrm{M}}^{\mathrm{ij}} = \left(\frac{\mathrm{E}^2 + \mathrm{H}^2}{8\pi\mathrm{c}}\right)\frac{\tilde{\mathrm{u}}^\mathrm{i}\tilde{\mathrm{u}}^\mathrm{j}}{c^2} \ll \frac{\mathrm{E}^2 + \mathrm{H}^2}{4\pi\mathrm{c}} = \mathrm{S}^{\mathrm{ij}} \tag{46}$$

For such fields we are left with the original Minikowski energy-momentum tensor and its conservation equations: $\partial_\mu \widetilde{\mathrm{T}}^{\mu\nu} \to \partial_\mu \mathrm{T}^{\mu\nu} = 0$.

The inertial forces as centrifugal and Coriolis's terms emerge straightforwardly from Eq.45 in cylindrical coordinates:

$$\frac{\partial \tilde{u}_r}{\partial t} + \tilde{u}_r\frac{\partial \tilde{u}_r}{\partial r} + \frac{\tilde{u}_\theta}{r}\frac{\partial \tilde{u}_r}{\partial \theta} + \tilde{u}_z\frac{\partial \tilde{u}_z}{\partial z} - \frac{\tilde{u}_\theta^2}{r} = \frac{1}{\rho}\mathrm{f}_\mathrm{r} \tag{47}$$

$$\frac{\partial \tilde{u}_\theta}{\partial t} + \tilde{u}_r\frac{\partial \tilde{u}_\theta}{\partial r} + \frac{\tilde{u}_\theta}{r}\frac{\partial \tilde{u}_\theta}{\partial \theta} + \tilde{u}_z\frac{\partial \tilde{u}_\theta}{\partial z} + \frac{\tilde{u}_r\tilde{u}_\theta}{r} = \frac{1}{\rho}\mathrm{f}_\theta \tag{48}$$

$$\frac{\partial \tilde{u}_z}{\partial t} + \tilde{u}_r\frac{\partial \tilde{u}_z}{\partial r} + \frac{\tilde{u}_\theta}{r}\frac{\partial \tilde{u}_z}{\partial \theta} + \tilde{u}_z\frac{\partial \tilde{u}_z}{\partial z} = \frac{1}{\rho}\mathrm{f}_\mathrm{z} \tag{49}$$

Where

$$\mathbf{f} = \frac{\mathbf{1}}{\mathbf{4\pi}}[\mathrm{E}\nabla\cdot\mathrm{E} + \mathrm{H}\nabla\cdot\mathrm{H} + \mathrm{E}\times(\nabla\times\mathrm{E}) + \mathrm{H}\times(\nabla\times\mathrm{H})]$$

Eq.47-49 include EM centrifugal force EM Coriolis force and 'EM material derivative' and also EM shear forces. These EM inertial forces are algebraic function of EM density and EM velocity and have the same structure as their parallel fluid/mechanical inertia terms and also the 'material derivative'.

VI. PROVING: MAXWELL'S EQUATIONS AND EM CONSERVATION EQUATIONS ARE EQUIVALENT

This equivalence is especially important to find the extension of Maxwell's equation which take in account the inertia of the EM fields (section VII).

We will prove the equivalence in two forms, the vector and tensorial form. Starting with the vector form, where the connection to the physical picture is clearer.

The energy-momentum conservation equations, without charges are:

$$\partial_\mu \mathrm{T}^{\mu\nu} = 0 \tag{50}$$

The three equations $\partial_\mu \mathrm{T}^{\mu\mathrm{i}} = 0$ in vector form which is non relativistic, meaning $\gamma \approx 1$, become:

$$\frac{\partial(\mathbf{E}\times\mathbf{H})}{4\pi c\partial t} = \frac{1}{4\pi}[\mathbf{E}\nabla\cdot\mathbf{E} + \mathbf{H}\nabla\cdot\mathbf{H} + \mathbf{E}\times(\nabla\times\mathbf{E}) + \mathbf{H}\times(\nabla\times\mathbf{H})] \tag{51}$$

Or

$$\dot{\mathbf{p}} = \frac{\mathbf{1}}{\mathbf{4\pi}}[\mathbf{E}\nabla\cdot\mathbf{E} + \mathbf{H}\nabla\cdot\mathbf{H} + \mathbf{E}\times(\nabla\times\mathbf{E}) + \mathbf{H}\times(\nabla\times\mathbf{H})] \tag{52}$$

The first equation $\partial_\mu \mathrm{T}^{\mu 0}$ is

$$\frac{\partial(\mathrm{E}^2 + \mathrm{H}^2)}{8\pi c\partial t} = \nabla\cdot\ \left(\frac{\mathbf{E}\times\mathbf{H}}{4\pi}\right) \tag{53}$$

divided by c can be written: $\dot{\tilde{\rho}} = \nabla \cdot\ \tilde{\mathbf{p}}$.

The vector equation Eq.49 and the scalar equation Eq.50 are equivalent to Maxwell's equations in vector form. To prove it we just rewrite the terms:

$$\begin{aligned}\mathbf{E}(\nabla\cdot\mathbf{E}) + \mathbf{H}\nabla\cdot\mathbf{H} + \mathbf{E}\times(\nabla\times\mathbf{E} + \frac{\partial\mathbf{H}}{c\partial t})\\ \mathbf{H}\times(\nabla\times\mathbf{H} - \frac{\mathbf{1}}{\mathbf{c}}\frac{\partial\mathbf{E}}{\partial\mathbf{t}}) = \mathbf{0}\end{aligned} \tag{54}$$

$$\mathbf{E}\cdot(\nabla\times\mathbf{E} + \frac{\partial\mathbf{H}}{c\partial t}) + \mathbf{H}\cdot(\nabla\times\mathbf{H} - \frac{\partial\mathbf{E}}{c\partial t}) = 0 \tag{55}$$

Beginning with the assumption that Maxwell equations are fulfilled, we get each term in the brackets is null, therefore the conservation equations are fulfilled. In the other direction - assuming that the conservation equations, Eq.51 and Eq.52, are fulfilled for all values of H and E, this can be satisfied only if each bracketed term is null separately. Each bracketed term is one of the four Maxwell equations, and since all have to be satisfied together, this gives exactly Maxwell's equations set. With this we proved the equivalence in both directions in vector form.

The equivalence also holds in tensorial form of Maxwell's equations:

$$\begin{cases}\partial_\mu \mathrm{F}^{\mu\nu} = 0\\ \partial_\mu \mathfrak{F}^{\mu\nu} = 0\end{cases} \tag{56}$$

and EM conservation equations in tonsorial form:

$$\begin{cases}\partial_\mu \mathrm{T}^{\mu\nu} = 0\\ \partial_\mu \mathfrak{F}^{\mu\nu} = 0\end{cases} \tag{57}$$

Eq. 53 and Eq. 54 mean that $\partial_\mu \mathrm{F}^{\mu\nu} = 0$ is equivalent to $\partial_\mu \mathrm{T}^{\mu\nu} = 0$ in conjunction with $\partial_\mu \mathfrak{F}^{\mu\nu} = 0$.

If we would work with the four-potentials A^μ, we would only need four equations. But, since we are working with six unknown functions $\mathbf{E}$ and $\mathbf{H}$ of $\mathrm{F}^{\mu\nu}$ defined as $\mathrm{F}^{\mu\nu} \equiv \partial^\mu \mathrm{A}^\nu - \partial^\nu \mathrm{A}^\mu$ as a four-rotor of A^μ, we must add these four equations $\partial_\mu \mathfrak{F}^{\mu\nu} = 0$, which can be understood as mathematical constrain, to reduce the six degrees of freedom to four.

$$\partial_\mu \mathfrak{F}^{\mu\nu} \equiv \partial_\mu[\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\mathrm{F}_{\rho\lambda}] = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(\partial_\mu\partial_\alpha \mathrm{A}_\beta - \partial_\mu\partial_\beta \mathrm{A}_\alpha) = 0$$

It is null mathematically, since we sum over a symmetrical partial derivatives $\partial_\mu\partial_\alpha = \partial_\alpha\partial_\mu$ and the anti-symmetry tensor $\epsilon^{\mu\nu\alpha\beta}$.

Now it is left to prove that $\partial_\mu \mathrm{T}^\mu_\nu = 0$ is equivalent to $\partial_\mu \mathrm{F}^\mu_\nu = 0$. We begin with this algebraic connection:

$$4\pi \mathrm{T}^{\mu\nu} = \mathrm{F}^\mu_\lambda \mathrm{F}^{\lambda\nu} + \mathfrak{F}^\mu_\lambda \mathfrak{F}^{\lambda\nu} = \mathrm{F}^\mu_\lambda \mathrm{F}^{\lambda\nu} - \frac{1}{4}\mathrm{F}^{\lambda\gamma}\mathrm{F}_{\lambda\gamma}\mathrm{g}^{\mu\nu} \tag{58}$$

Its four divergence is:

$$4\pi\partial_\mu \mathrm{T}^\mu_\nu = (\partial_\mu \mathrm{F}^{\mu\lambda})\mathrm{F}_{\lambda\nu} + \mathrm{F}^{\mu\lambda}\partial_\mu \mathrm{F}_{\lambda\nu} - \frac{1}{4}\partial_\nu(\mathrm{F}^{\lambda\gamma}\mathrm{F}_{\lambda\gamma}) \tag{59}$$

The first direction of the proof: Since $\partial_\mu \mathrm{F}^{\mu\lambda} = 0$ the first term in Eq.56 is null, we need to prove that the second and third terms cancel each other. For this we change the dummy summing indexes μ and λ : $\mathrm{F}^{\mu\lambda}\partial_\mu \mathrm{F}_{\lambda\nu} = \mathrm{F}^{\lambda\mu}\partial_\lambda \mathrm{F}_{\mu\nu}$, therefore we can write $\mathrm{F}^{\mu\lambda}\partial_\mu \mathrm{F}_{\lambda\nu} = \frac{1}{2}\mathrm{F}^{\lambda\mu}(\partial_\mu \mathrm{F}_{\lambda\nu} + \partial_\lambda \mathrm{F}_{\mu\nu})$, using the conjunction equations $\partial_\mu \mathfrak{F}^{\mu\nu} = 0$ in this form $\partial_\mu \mathrm{F}_{\lambda\nu} + \partial_\lambda \mathrm{F}_{\nu\mu} + \partial_\nu \mathrm{F}_{\mu\lambda} = 0$, and installing them in Eq.56 we get $F^{\mu\lambda}\partial_\nu \mathrm{F}_{\mu\lambda} = \frac{1}{4}\partial_\nu(\mathrm{F}^{\mu\lambda}\mathrm{F}_{\mu\lambda})$, Therefore the second and third terms in Eq.56 cancel out. Therefore $\partial_\mu \mathrm{T}^\mu_\nu = 0$ are fulfilled.

In the other direction: We start with the conjugate equations $\partial_\mu \mathfrak{F}^{\mu\lambda} = 0$, these equations can also be written as $\partial_\mu \mathrm{F}_{\lambda\nu} + \partial_\lambda \mathrm{F}_{\nu\mu} + \partial_\nu \mathrm{F}_{\mu\lambda} = 0$. Multiplying these equations by $F^{\mu\lambda}$ and changing the summation indexes we get: $F^{\mu\lambda}\partial_\nu \mathrm{F}_{\mu\lambda} = \frac{1}{4}\partial_\nu(\mathrm{F}^{\mu\lambda}\mathrm{F}_{\mu\lambda})$, this cancels out the second and third terms in Eq.56. It is assumed that the conservation equations are fulfilled $\partial_\mu \mathrm{T}^\mu_\nu = 0$ or $(\partial_\mu \mathrm{F}^{\mu\lambda})\mathrm{F}_{\lambda\nu} + \mathrm{F}^{\mu\lambda}\partial_\mu \mathrm{F}_{\lambda\nu} - \frac{1}{4}\partial_\nu(\mathrm{F}^{\lambda\gamma}\mathrm{F}_{\lambda\gamma}) = 0$, therefore, we are left with these equations: $(\partial_\mu \mathrm{F}^{\mu\lambda})\mathrm{F}_{\lambda\nu} = 0$. This equation is true for all values of $\mathrm{F}_{\lambda\nu}$ only if $\partial_\mu \mathrm{F}^{\mu\lambda} = 0$, which means that the four Maxwell equations must be fulfilled. Thus we have proved the equivalence in both directions.

The energy-momentum conservation equations when charges are not present become:

$$\partial_\mu \mathrm{T}^{\mu\nu} = 0 \tag{60}$$

are equivalent to Maxwell's equations without charge currents

$$\partial_\mu \mathrm{F}^{\mu\nu} = 0 \tag{61}$$

The only difference between Maxwell's equations and the EM energy-momentum conservation equations are the physical quantities that are used. Maxwell's equations use E and H while the EM conservation equations are using quantities which are a function of E and H and have analogue quantities in fluid mechanics, like the momentum density $\tilde{\mathrm{p}} = \frac{\mathbf{E}\times\mathbf{H}}{4\pi\mathrm{c}}$ and the energy density $\tilde{\rho} = \frac{\mathrm{E}^2+\mathrm{H}^2}{8\pi c^2}$. Although Maxwell's equations are equivalent to the conservation equations, the Maxwell's equations are linear as function of the fields E and H while the conservation equations are non-linear in E and H since the momentum and density are non linear functions of E and H. The advantage of the conservation equations presentation is the clear connection to the mechanical phenomena, especially important when momentum and energy can transform from EM fields to matter and vice versa, as in the case when a charged particle radiates.

This equivalence also helps to understand what is the role of Minikowski's tensor in electrodynamics. It also clears why we can not use the energy-momentum tensor's components as proper definition for the EM energy-momentum density, since Maxwell's stresses are essential for this equivalence and can never be removed.

VII. FINDING THE EXTENDED MAXWELL'S EQUATIONS WHICH ARE EQUIVALENT TO THE EXTENDED CONSERVATION EQUATIONS

The new conservation equations Eq.45 are the extended motion equations of EM fields. These equations become important especially when intense EM fields are involved. As a general rule we can say that this is the case when the flux terms $\frac{\tilde{\rho}}{c}\tilde{u}^i\tilde{u}^j$ cannot be neglected when their magnitude is in the same order of magnitude as the stress terms. In the last section we proved that the original energy-momentum conservation equations are equivalent to the original Maxwell's equations. This equivalence must hold for the extended conservation equations as well, meaning that there is an extension to Maxwell's equations which makes them equivalent to Eq.45.

To find this extension we start by writing Eq.45 in vector form. The first row is:

$$\frac{\partial}{\partial t}(\tilde{\rho} - \frac{1}{c^2}\tilde{\rho}\tilde{u}^2) + \nabla \cdot (\tilde{\rho}\tilde{\mathbf{u}}) = 0 \tag{62}$$

The structure of this equation is known as energy-mass conservation equation, excluding the term $\frac{1}{c^2}\tilde{\rho}\tilde{u}^2$. Since this term is proportional to $\frac{1}{c^2}$ it becomes significant only when $\tilde{u} \to c$, so in most of the cases it can be neglected. We will get into more details about this point when EM waves and other examples will be introduced.

The other three rows in details can be written as:

$$\partial_t\left(\rho\tilde{u}^j\right) - \partial_i\left(-\frac{1}{4\pi c}\left(E^iE^j + H^iH^j\right) + \left(\frac{E^2+H^2}{8\pi c}\right)\delta^{ij} - \frac{\tilde{\rho}}{c}\tilde{u}^i\tilde{u}^j\right) = 0 \tag{63}$$

After taking the derivatives and rearranging the three equations together we get:

$$\partial_t\left(\rho\tilde{\mathbf{u}}\right) + \tilde{\mathbf{u}}\nabla\cdot\left(\rho\tilde{\mathbf{u}}\right) + \rho\left(\tilde{\mathbf{u}}\cdot\nabla\right)\tilde{\mathbf{u}} = \mathbf{f} \tag{64}$$

Where $\mathbf{f}$ represent the inner forces densities of the EM fields, the same as the non extended equation have:

$$\mathbf{f} = \frac{\mathbf{1}}{\mathbf{4}\pi}[E\nabla\cdot E + H\nabla\cdot H + E\times(\nabla\times E) + H\times(\nabla\times H)] \tag{65}$$

Insert the energy-mass conservation equation Eq.59 in the case $|\tilde{u}| \ll c$ into Eq.61 above :

$$\partial_t\tilde{\mathbf{u}} + \tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}} = \frac{1}{\tilde{\rho}}\mathbf{f} \tag{66}$$

These equations have the same structure as the Euler equations if we replace the tilde-EM quantities, with the non-tilde mechanical quantities: $\tilde{\mathbf{u}} \to \mathbf{u}$ and $\tilde{\rho} \to \rho$. The EM 'material derivative': $\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}$ is the EM analog to 'material derivative' $\mathbf{u}\cdot\nabla\mathbf{u}$.

We can find the extended Maxwell's equations by rewriting Eq.61 as:

$$\partial_t\tilde{\mathbf{p}} - \mathbf{f} + \tilde{\mathbf{p}}\nabla\cdot\tilde{\mathbf{u}} + \tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{p}} = \mathbf{0} \tag{67}$$

Since $\partial_t\tilde{\mathbf{p}} = \frac{1}{4\pi c}(\mathbf{H}\times(-\frac{\partial\mathbf{E}}{\mathbf{c}\partial\mathbf{t}}) + \mathbf{E}\times\frac{\partial\mathbf{H}}{\mathbf{c}\partial\mathbf{t}})$ the last equation, as function of $\mathbf{E}$ and $\mathbf{H}$, can be written as:

$$\mathbf{H}\times(-\frac{\partial\mathbf{E}}{c\partial t} + \nabla\times\mathbf{H}) - \frac{1}{c}(\mathbf{H}\times\mathbf{E})\nabla\cdot\tilde{\mathbf{u}} + (\tilde{\mathbf{u}}\cdot\nabla)(\mathbf{E}\times\mathbf{H}) \\ -c\mathbf{E}\nabla\cdot\mathbf{E} - c\mathbf{H}\nabla\cdot\mathbf{H} + \mathbf{E}\times(\frac{\partial\mathbf{H}}{c\partial t} + \nabla\times\mathbf{E}) = 0 \tag{68}$$

Taking the derivative $(\tilde{\mathbf{u}}\cdot\nabla)$ we get:

$$(\tilde{\mathbf{u}}\cdot\nabla)(\mathbf{E}\times\mathbf{H}) = [\tilde{\mathbf{u}}\cdot\nabla\mathbf{E}]\times\mathbf{H} + \mathbf{E}\times[\tilde{\mathbf{u}}\cdot\nabla\mathbf{H}]$$

Inserting in Eq.65 we get:

$$-c\mathbf{E}\nabla\cdot\mathbf{E} - c\mathbf{H}\nabla\cdot\mathbf{H} + \\ \mathbf{E}\times(\frac{\partial\mathbf{H}}{c\partial t} + \nabla\times\mathbf{E}) + \mathbf{H}\times(-\frac{\partial\mathbf{E}}{c\partial t} + \nabla\times\mathbf{H}) + \\ \frac{1}{c}(\mathbf{E}\times\mathbf{H})\nabla\cdot\tilde{\mathbf{u}} + \frac{\mathbf{1}}{\mathbf{c}}[\tilde{\mathbf{u}}\cdot\nabla\mathbf{E}]\times\mathbf{H} + \frac{\mathbf{1}}{\mathbf{c}}\mathbf{E}\times[\tilde{\mathbf{u}}\cdot\nabla\mathbf{H}] = \mathbf{0} \tag{69}$$

Rewriting the extended term:

$$\frac{1}{c}\mathbf{E}\times[\tilde{\mathbf{u}}\cdot\nabla\mathbf{H}] = \frac{\mathbf{1}}{\mathbf{c}}\mathbf{H}\times[\mathbf{E}\frac{\mathbf{2E}\cdot(\nabla\times\mathbf{H})}{E^2+H^2}] \tag{70}$$

and rearranging the brackets terms as follows:

$$\mathbf{H}\times(-\frac{\partial\mathbf{E}}{c\partial t} + \nabla\times\mathbf{H} - \tfrac{1}{c}\mathbf{E}\nabla\cdot\tilde{\mathbf{u}} - \tfrac{\mathbf{1}}{\mathbf{c}}\tilde{\mathbf{u}}\cdot\nabla\mathbf{E} + \\ \frac{1}{c}\mathbf{E}[\frac{\mathbf{2E}\cdot(\nabla\times\mathbf{H})}{E^2+H^2}]) + \mathbf{E}\times(\frac{\partial\mathbf{H}}{\mathbf{c}\partial\mathbf{t}} + \nabla\times\mathbf{E}) \\ -c\mathbf{E}\nabla\cdot\mathbf{E} - c\mathbf{H}\nabla\cdot\mathbf{H} = 0 \tag{71}$$

In order for these equations to be satisfied for all values of $\mathbf{E}$ and $\mathbf{H}$ each bracket term must always be null. This gives the Extended Maxwell's equations in vacuum:

$$\frac{\partial\mathbf{E}}{c\partial t} - \nabla\times\mathbf{H} + \tfrac{1}{c}\mathbf{E}\nabla\cdot\tilde{\mathbf{u}} + \tfrac{\mathbf{1}}{\mathbf{c}}\tilde{\mathbf{u}}\cdot\nabla\mathbf{E} + \tfrac{1}{c}\mathbf{E}[\tfrac{\mathbf{2E}\cdot(\nabla\times\mathbf{H})}{E^2+H^2}] = \mathbf{0} \\ \frac{\partial\mathbf{H}}{c\partial t} + \nabla\times\mathbf{E} = 0 \\ \nabla\cdot\mathbf{E} = 0 \\ \nabla\cdot\mathbf{H} = 0 \tag{72}$$

When external charge density and/or current density are involved, the extended Maxwell's equations become:

$$\frac{\partial\mathbf{E}}{c\partial t} - \nabla\times\mathbf{H} = \tfrac{4\pi}{c}\mathbf{j} - \tfrac{\mathbf{E}}{\mathbf{c}}\nabla\cdot\tilde{\mathbf{u}} - \tfrac{\tilde{\mathbf{u}}}{\mathbf{c}}\cdot\nabla\mathbf{E} - \tfrac{\mathbf{E}}{c}[\tfrac{\mathbf{2E}\cdot(\nabla\times\mathbf{H})}{E^2+H^2}] \\ \frac{\partial\mathbf{H}}{c\partial t} + \nabla\times\mathbf{E} = 0 \\ \nabla\cdot\mathbf{E} = 4\pi\rho \\ \nabla\cdot\mathbf{H} = 0 \tag{73}$$

The new terms are proportional to $\frac{\tilde{u}}{c}$, therefore these terms are not relevant for cases when $\frac{|\tilde{u}|}{c} \ll 1$, in these cases the extended equations reduce to the regular Maxwell's equations.

Now we are looking for the general extended EM wave equations. We do the similar procedure done on Maxwell's equation to get the EM waves. First we take the rotor of the first modified equation of Eq.73 and taking the partial time derivative on its second equation, then installing the second in the first, and using Eq.73 fourth equation the $\nabla \cdot \mathbf{H} = \mathbf{0}$, we get:

$$\nabla^2\mathbf{H} - \frac{\partial^2\mathbf{H}}{c^2\partial t^2} = \frac{4\pi}{c}\nabla \times [\frac{1}{4\pi}\mathbf{E}\nabla \cdot \tilde{\mathbf{u}} + \frac{\mathbf{1}}{\mathbf{4\pi}}\tilde{\mathbf{u}} \cdot \nabla\mathbf{E} + \frac{1}{4\pi}\mathbf{E}[\frac{\mathbf{2E}\cdot(\nabla\times\mathbf{H})}{\mathrm{E}^2+\mathrm{H}^2}] + \mathbf{j}] \tag{74}$$

We define an effective EM current density

$$\tilde{\mathbf{J}}_{\mathbf{EM-Field}} = \frac{\mathbf{1}}{\mathbf{4\pi}}\mathbf{E}\nabla\cdot\tilde{\mathbf{u}} + \frac{\mathbf{1}}{\mathbf{4\pi}}\tilde{\mathbf{u}}\cdot\nabla\mathbf{E} + \frac{1}{4\pi}\mathbf{E}[\frac{\mathbf{2E}\cdot(\nabla\times\mathbf{H})}{\mathrm{E}^2+\mathrm{H}^2}]$$

which is made of pure EM fields, Eq.74 can be written as:

$$\nabla^2\mathbf{H} - \frac{\partial^2\mathbf{H}}{c^2\partial t^2} = \frac{4\pi}{c}\nabla \times [\tilde{\mathbf{J}}_{\mathbf{EM-Field}} + \mathbf{j}] \tag{75}$$

The electric field's extended wave equation is derived similarly, first we take the rotor of the second term of Eq.73 and then taking the partial time derivative $\frac{\partial}{c\partial t}$ of the first equation of Eq.73. Installing the second into the first using the third equation $\nabla\cdot\mathbf{E} = 4\pi\rho_{\mathrm{e}}$ of Eq.73, after rearranging becomes:

$$\nabla^2\mathbf{E} - \frac{\partial^2\mathbf{E}}{c^2\partial t^2} = -\frac{4\pi}{c}\frac{\partial}{\partial t}[\tilde{\mathbf{J}}_{\mathbf{EM-Field}} + \mathbf{j}] - \mathbf{4\pi}\nabla\rho_{\mathbf{e}} \tag{76}$$

When $\tilde{\mathbf{J}}_{\mathbf{EM-Field}} \approx \mathbf{0}$ we call it vacuum, since the EM waves equations become Maxwellian wave equations. We note that vacuum means no charge particle and no strong EM fields are present.

An interesting phenomena happens when a wave encounters a volume where intense electric or magnetic field are present $\tilde{\mathbf{J}}_{\mathbf{EM-Field}} \neq \mathbf{0}$, meaning, the wave equations are not Maxwellian any more, in such a case the wave front is distorted. According to Huygens' principle the light direction will change or i.e bend.

An example of such phenomena should exist near pulsars, or strong electric field that exist near charged black holes. In such cases the EM wave will be impacted directly by the nonlinear terms in the wave equations. The solution for such cases is elaborate since it demands a solution of the non-linear wave equations Eq.72 and/or Eq.73, even the approximation is very long for the scope of this paper.

Now we can resolve the problem (see Section V) of the charged cylindrical electrode inside a long solenoid . (EM fields from a material source are always weak, otherwise the fields stresses on the material will destroy the material, also if $\mathbf{E}$ field is higher than $|\mathbf{E}| > \mathbf{10^{8-9}V/m}$ field emission will occur).

First we will show that using the regular Maxwell equations brings to a conflict. We start by assuming Maxwell's equations and Minikowski's motion equations are a complete and full representation of this case. The electrostatic field of the long charged rod, according to Maxwell's equations, is $\mathbf{E} = \frac{\lambda}{\mathbf{r}}\hat{\mathbf{r}}$ where the constant $\lambda = \frac{\mathrm{V_{a-b}}}{ln(a/b)}$. Inside the long solenoid's magnetic field, according to Maxwell' equations, is $\mathbf{H} = \mathrm{H_0}\hat{\mathrm{z}}$.

Although the electric and the magnetic fields are weak, the absence of centrifugal force is still evident, because according to Poynting, the EM momentum is on theta direction: $\tilde{\mathbf{p}} = \frac{\lambda\mathrm{H_0}}{\mathbf{4\pi cr}}\hat{\theta}$, this can also be written $\tilde{\mathbf{u}} \equiv \frac{\tilde{\mathbf{p}}}{\tilde{\rho}} = \frac{\lambda\mathrm{H_0}}{\mathbf{4\pi c\tilde{\rho}r}}\hat{\theta} = \frac{\mathbf{2c}\lambda\mathrm{H_0}}{(\frac{\lambda^2}{\mathbf{r}}+\mathrm{H_0}^2\mathrm{r})}\hat{\theta} = (\mathbf{0}, \tilde{\mathbf{u}}_\theta(\mathbf{r}), \mathbf{0})$. Therefore, this term $\tilde{\rho}\frac{\tilde{u}_\theta^2}{r}$ is not null for this situation, it has the units of force density and has the same form as centrifugal force of fluids. Inserting the rod and solenoid fields into the momentum motion equation of Minikowski Eq.35:

$$\dot{(\tilde{\rho}\tilde{u}^j)} + \partial_i s^{ij} = 0$$

Gives $0 = 0$, meaning these fields are an exact solution of Eq.35 as assumed, but the term $\tilde{\rho}\frac{\tilde{u}_\theta^2}{r}$ does not appear in Eq.35 in any stage and we are left with the following conundrum: If we assume these equations express the EM momentum propagation in full and we accept Poynting Theorem, than we might consider the force term $\tilde{\rho}\frac{\tilde{u}_\theta^2}{r}$, although it is not null, just as a virtual centrifugal force. Otherwise, if we add this term as inertial density force into Eq.35 we get $\tilde{\rho}\frac{\tilde{u}_\theta^2}{r} = 0$ which is false by Poynting, therefore Eq.35 are not fulfilled.

This contradiction between Poynting force and the Minikowski EM momentum motion equations does not exist if we take the extended motion equation of EM momentum Eq.47. Inserting the fields above, as a first order solution, into Eq. 72 and using Eq. 47 we get:

$$-\tilde{\rho}\frac{\tilde{u}_\theta^2}{r} = \frac{1}{4\pi}\mathbf{H}' \times (\nabla \times \mathbf{H}') \tag{77}$$

Although rotor of a constant is null, it is not the case here since $\mathbf{H}' = \mathbf{H_{Maxwell}} + \delta\mathbf{H}$, where $\mathbf{H_{Maxwell}} = \mathrm{H_0}\hat{\mathrm{z}}$. The $\delta\mathbf{H}$ is the additional field that arises from the solution of the first row of Eq.72 and $|\delta\mathbf{H}| << |\mathbf{H_{Maxwell}}|$.

$$\mathbf{H}'\times(\nabla\times\mathbf{H}') = -\frac{\mathbf{1}}{\mathbf{c}}\mathbf{H}'\times\mathbf{E}\nabla\cdot\tilde{\mathbf{u}}' - \frac{\mathbf{1}}{\mathbf{c}}\mathbf{H}'\times\tilde{\mathbf{u}}'\cdot\nabla\mathbf{E} - \frac{\mathbf{H}'\times\mathbf{E}}{\mathrm{c}}[\frac{\mathbf{2E}\cdot(\nabla\times\mathbf{H}')}{\mathrm{E}^2+\mathrm{H}'^2}]$$

We should notice that $\tilde{\mathbf{u}}' = \frac{\mathbf{2E}\times\mathbf{H}'}{\mathbf{E^2}+\mathbf{H'^2}}$.

In this example we have cylindrical symmetry, which means that H' and u' can only be a function of r or z, but not a function of theta. The first term on the right is null, since $\nabla \cdot \tilde{\mathbf{u}}' = \frac{\mathbf{1}}{\mathrm{r}}\frac{\partial\tilde{\mathbf{u}}'_\theta}{\partial\theta} = \mathbf{0}$.

Using Eq.70: $\frac{1}{c}\mathbf{H'}\times[\mathbf{E}\frac{\mathbf{2E}\cdot(\nabla\times\mathbf{H'})}{\mathrm{E}^2+\mathrm{H'}^2}] = \frac{\mathbf{1}}{\mathbf{c}}\mathbf{E}\times[\tilde{\mathbf{u}}\cdot\nabla\mathbf{H'}]$. Therefore, $\tilde{\mathbf{u}}\cdot\nabla\mathbf{H'}=\mathbf{0}$

We are left with:

$$\mathbf{H'}\times(\nabla\times\mathbf{H'}) = -\frac{\mathbf{1}}{\mathbf{c}}\mathbf{H'}\times\tilde{\mathbf{u}}'\cdot\nabla\mathbf{E} \quad (78)$$

Using the 'material derivative' in cylindrical coordinates we get:

$$-\frac{1}{c}(\mathbf{u'}\cdot\nabla)\mathbf{E} = -\frac{\mathbf{1}}{\mathbf{c}}(\tilde{\mathbf{u}}'_{\mathbf{r}}\partial_{\mathbf{r}}\mathbf{E}(\mathbf{r})_\theta - \frac{\mathbf{u'}_\theta\mathbf{E_r}}{\mathbf{c}\mathrm{r}} = -\frac{\mathbf{u'}_\theta\mathbf{E_r}}{\mathbf{c}\mathrm{r}}\hat{\theta} \quad (79)$$

Inserting Eq.79 on the right side of Eq.78 we get:

$$-\tilde{\rho}\frac{\tilde{u}'^2_\theta}{r}\hat{r} = \frac{1}{4\pi}\mathbf{H'}\hat{\mathbf{z}}\times(-\frac{\mathbf{u'}_\theta\mathbf{E_r}}{\mathbf{c}\mathrm{r}}\hat{\theta}) \quad (80)$$

Using the definition of the momentum on θ the momentum is $\tilde{p}'_\theta = \frac{\mathrm{H}'_z\mathbf{E_r}}{4\pi c}$, as seen both sides become the same and the conflict is gone. We see that by using the extended equations the centrifugal force is balanced by the 'EM field current density' term: $\frac{1}{4\pi c}\tilde{\mathbf{u}}'\cdot\nabla\mathbf{E}$.

By using the extended Maxwell equations and the extended EM momentum motion equation, instead of the regular Maxwell's equations and Minikowski's EM momentum motion equations, the contradiction with the Poynting force disappears.

VIII. THE LAGRANGIAN OF THE EXTENDED EQUATIONS

The Lagrangian of the extended equations must be a Lorentz invariant scalar, the natural candidate is

$$\mathcal{L}_{EE} = \widetilde{\rho}\mathrm{c}^2 = \frac{1}{8\pi}\sqrt{\left(\tfrac{1}{2}\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}\right)^2 + \left(\tfrac{1}{2}\mathrm{F}_{\mu\nu}\mathfrak{F}^{\mu\nu}\right)^2 + 2\mathrm{P}_{\mu\nu}\mathrm{P}^{\mu\nu}} \quad (81)$$

the action is:

$$\mathrm{S} = \int\widetilde{\rho}\mathrm{c}^2\sqrt{-\mathrm{g}}\mathrm{d}^4\mathrm{x}$$

To find the motion equations we take the variation of the action as function of $\mathrm{A}^{\mu\nu}$ and equating to zero i.e. finding the extramum of the action. Since the action is only depended on derivatives of the potential $\partial^\mu A^\nu$ and not on A^ν the functional derivative becomes:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)} = \partial_\mu\mathrm{G}^{\mu\nu} = 0 \quad (82)$$

These give the extended Maxwell's equations in covariant form, in details becomes:

$$\begin{cases}\partial_\mu\mathrm{F}^{\mu\nu} - \frac{1}{\mathrm{c}}\partial_\mu(\mathrm{F}^{\mu\rho}\tilde{\mathrm{u}}^\nu_\rho + \mathfrak{F}^{\mu\rho}\tilde{\mathrm{u}}^\nu_\rho) = 0 \\ \partial_\mu\mathfrak{F}^{\mu\nu} = 0\end{cases} \quad (83)$$

The first equation in Eq.83 can also be written as:

$$\mathbf{D}\cdot(\mathbf{F} - \frac{\mathbf{1}}{\mathbf{c}}\mathrm{F}\dot{\times}\tilde{\mathbf{u}} + \frac{\mathbf{1}}{\mathbf{c}}\mathfrak{F}\dot{\times}\tilde{\mathbf{u}}) = \mathbf{0}$$

A. A brief comparison to other nonlinear extensions of EM Lagrangians of extensions of Maxwell's equations

In 1936 Born and Infeld started their nonlinear Electrodynamics, which is an extension of Maxwell's equations. Their main motivation was to solve the self energy of a point charge. Lately, the interest in their theory was renewed due to investigations of the deeper relation between QED and string theory. The Born-Infeld Lagrangian is

$$\mathcal{L}_{BI} = b^2\left(1 - \sqrt{1 - \frac{1}{2b^2}\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu} - \left(\frac{1}{4b^2}\mathrm{F}_{\mu\nu}\mathfrak{F}^{\mu\nu}\right)^2}\right) \quad (84)$$

The added constant b is a parameter which measures the non-linearity of the theory. In the limit $b\to\infty$ the Lagrangian $\mathcal{L}_{BI}$ tends to the Maxwell's Lagrangian $\mathcal{L}_{Maxwell} = \frac{1}{2}\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}$. Comparing the Lagrangian density of the extended equation $\mathcal{L}_{EE}$ Eq.81 to Born-Infeld Lagrangian $\mathcal{L}_{BI}$ Eq.84 reveals a few differences, first $\mathcal{L}_{EE}$ does not need any new (adjustable) parameter b, another difference, which is a deeper argument for the correctness of this Lagrangian, is revealed by the energy-momentum tensor of the Born-Infeld field:

$$\begin{aligned}\mathrm{T}^{00} &= \mathrm{b}^2\left(\sqrt{1+\mathrm{b}^{-2}(\mathbf{D^2}+\mathbf{B^2})+\mathbf{b^{-4}}\left(\mathbf{D}\times\mathbf{B}\right)^{\mathbf{2}}}-1\right)\\ \mathrm{T}^{0\mathrm{k}} &= (\mathbf{D}\times\mathbf{B})^{\mathrm{k}}\\ \mathrm{T}^{\mathrm{kl}} &= \delta^{\mathrm{kl}}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{D}-\mathrm{T}^{00}) - (\mathrm{E}^{\mathrm{k}}\mathrm{D}^{\mathrm{l}}+\mathrm{H}^{\mathrm{k}}\mathrm{D}^{\mathrm{l}})\end{aligned} \quad (85)$$

Where

$$\begin{aligned}\mathrm{D} &= \frac{\mathbf{E}+\mathbf{b^{-2}}(\mathbf{E}\cdot\mathbf{B})\mathbf{B}}{\sqrt{1-\mathrm{b}^{-2}(\mathbf{E^2}-\mathbf{B^2})+\mathbf{b^{-4}}\left(\mathbf{E}\cdot\mathbf{B}\right)^{\mathbf{2}}}}\\ \mathrm{H} &= \frac{\mathbf{B}-\mathbf{b^{-2}}(\mathbf{E}\cdot\mathbf{H})\mathbf{B}}{\sqrt{1-\mathrm{b}^{-2}(\mathbf{E^2}-\mathbf{B^2})+\mathbf{b^{-4}}\left(\mathbf{E}\cdot\mathbf{B}\right)^{\mathbf{2}}}}\end{aligned} \quad (86)$$

We notice that even here, there are no momentum flux terms $\frac{\tilde{\mathrm{p}}^{\mathrm{k}}\tilde{\mathrm{p}}^{\mathrm{l}}}{\tilde{\rho}} = \tilde{\rho}\tilde{\mathrm{u}}^{\mathrm{k}}\tilde{\mathrm{u}}^{\mathrm{l}}$. These terms are present in T_{EE} and originate from the terms $2\mathrm{P}_{\mu\nu}\mathrm{P}^{\mu\nu}$ in $\tilde{\rho}$ which are missing in $\mathcal{L}_{BI}$. These arguments are true also for Euler-Heisenberg Lagrangian

$$\mathcal{L}_{HE} = \frac{\mathbf{E^2}-\mathbf{H^2}}{2} + \frac{1}{90\pi}\frac{\hbar c}{e^2}\frac{1}{{E_0}^2}\left[(\mathbf{E^2}-\mathbf{H^2})^{\mathbf{2}}+\mathbf{7}(\mathbf{E}\cdot\mathbf{H})^{\mathbf{2}}\right] \quad (87)$$

Eq.87 also uses constants, $\hbar$ and the electron charge e, that do not originate from 'pure' electrodynamics. In the absence of material, these constants have no place in the motion equations.

This, Probably, makes $\mathcal{L}_{EE}$ and its derivative equations, the most natural choice for nonlinear extension of Maxwell's equations, it may also be more suitable for QED and string theory. In the next articles we will investigate these other issues concerning these extended equations.

IX. RESOLVING THE RADIATION PROBLEM

In this section we will use the new extended motion equations to settle the radiation problem. First let us introduce the problem, which is dealing with the path of an accelerated charged particle, which takes in account radiation and its recoil force on the particle.

The simplest motion equation for a charged particle is Newton's equation, with only one EM force term $q\mathbf{E}$:

$$m\mathbf{a} = \mathbf{qE} \tag{88}$$

This motion equation does not take into account the recoil force on the accelerating particle caused by the emitting electromagnetic radiation from the charged particle. To clear this point, lets look at accelerated neutral particle with mass:

$$m_{neutral-particle} \equiv m_{mechanical} \equiv m_0$$

the charged particle or body which has exactly the same total mass

$$m_{neutral-particle} \equiv m_{charged-particle}$$

were total means:

$$m_{total} \equiv m_{charged-particle} = m_0 + m_{charge} + m_{field}$$

were

$$m_{charge} = \Sigma m_{electron}$$

and

$$m_{field} = \frac{\mathrm{E}^2}{8\pi c^2}$$

is the mass of the EM field.

According to Newton's equation above, we have the same acceleration under the same force, meaning both particles will gain the same energy over the same distance. On the other hand we know, a charged particle under acceleration radiates EM waves, these waves have momentum, so why is this momentum balance not represented in the motion equations?

This dilemma is known as 'radiation reaction force problem' since 1872, also known as radiation reaction or 'Abraham–Lorentz force problem'. We bring a brief summery of its treatment according to Lorentz and Abraham (Jackson 22-82). The radiation reaction force was represented by an additional term in Newton equation Eq.88:

$$m\mathbf{a} = q\mathbf{E} + \frac{\mathbf{2e^2}}{\mathbf{3c^3}}\dot{\mathbf{a}} \tag{89}$$

The added term of Abraham -Lorentz $\frac{2e^2}{3c^3}\dot{\mathbf{a}}$ in Eq.89 represents the force which gives the total radiation power of the accelerating charge. The total power of accelerating charge is given by Larmor's equation which is : $\mathrm{P} = \frac{2\mathrm{e}^2}{3\mathrm{c}^3}\mathrm{a}^2$. The connection between power and force is $\mathrm{W} = \mathbf{v}\cdot\mathbf{F}$, therefore, we can extract the force related to the Larmor radiation power by: $\mathrm{W} = \frac{2\mathrm{e}^2}{3\mathrm{c}^3}\mathbf{v}\cdot\dot{\mathbf{a}} = \frac{\mathbf{2e^2}}{\mathbf{3c^3}}\mathbf{v}\cdot\ddot{\mathbf{v}} = \frac{\mathbf{2e^2}}{\mathbf{3c^3}}[\frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{v}\cdot\dot{\mathbf{v}}) - \dot{\mathbf{v}}^\mathbf{2}]$. The term $\frac{d}{dt}\int(\mathbf{v}\cdot\dot{\mathbf{v}})\mathbf{d^3x}$ for periodic motion averages to zero, therefore we are left with the Larmor force $\mathrm{F}_{\mathrm{Larmor}} = \frac{2\mathrm{e}^2}{3\mathrm{c}^3}\dot{\mathrm{a}}$.

Symbol $t_0 = \frac{2e^2}{3mc^3}$ Eq.89 becomes:

$$m\mathbf{a} = \mathbf{mt_0}\dot{\mathbf{a}} + \mathbf{F}_{\mathrm{ext}}$$

We have a simple deferential equation which can be integrated to give:

$$m\mathbf{a} = \frac{\mathbf{1}}{\mathbf{t_0}}\int_{\mathbf{t}}^{\infty}\exp\left(-\frac{\mathbf{t'}-\mathbf{t}}{\mathbf{t_0}}\right)\mathbf{F}_{\mathrm{ext}}(\mathbf{t'})\,\mathbf{dt'}$$

This solution has a noticeable problem, the integral starts from time $t = t_0 \neq 0$ and not from the present $t = 0$ as we would expect. This means that future values of the force affect the acceleration of the particle in the present. If it was the only problem, we could remove it by ignoring this relatively small term, but there are other more fundamental problems, we will list only a few of them.

Starting with the lack of force term that should represent radiation force when the particle is in constant acceleration. We expect such term to exist since Larmor radiation equation depends on $\mathbf{a}$ and not on $\dot{\mathbf{a}}$ as in Eq.89. Other terms we expect to exist are correction terms of the external electric force $q\mathbf{E}$ arising from the relative velocity and relative acceleration of the moving particle compared to the external electric source. Another term that should exist is a term representing the outgoing radiation momentum per unit time and its directional reaction on the particle momentum. Another problem is, why the reaction force of the radiation is on the direction of $\dot{\mathbf{a}}$, and not on any other direction?

Before we start with resolving the problem, a few words on the root of this problem. We all know that radiation is an EM phenomena and EM fields have their own equation of motion, which are not represented in the motion equation of Abraham–Lorentz Eq.89.

This classic problem of accelerated charged particle and other open EM problems, are discussed by Feynman in his Lecture 28-1 as was presented in the introduction.

Since EM fields are involved, we must consider their dynamics (EM momentum flow) and their coupling to the massive core's dynamics (mechanical momentum flow).

When a charged particle is accelerated we are dealing with a non trivial system, which includes the external EM fields, the charged particle's EM fields and its mechanical core mass. The equations that describe all these interactions are the extended conservation equations coupled to the mechanical conservation equation. The coupling of the mechanics and the electrodynamics is done by adding the energy-momentum conservation equations of EM fields and the energy-momentum conservation of mechanics/fluids. Since we are using six mathematical function fields **E** and **H** and not the original four mathematical function fields A^μ, we need the conjugate equations $\partial_\mu \mathfrak{F}^{\mu\nu} = 0$. The equations that represent the complete motion equations of charged particles with their fields is therefore:

$$\begin{cases} \partial_\mu T^{\mu\nu}_{mechanics} + \partial_\mu \tilde{\mathrm{T}}^{\mu\nu}_{EM} = 0 \\ \partial_\mu \mathfrak{F}^{\mu\nu} = 0 \end{cases} \tag{90}$$

These motion equations state that the addition of the mechanical and the EM energy-momentum are conserved together (as a sum) in any point in space and time. Therefore, solving one and than installing the solution in the other, is just an approximation.

The precise solution of Eq.90 when the EM fields are not known in advance or when the mechanical paths are not given, is extremely challenging, even for only one moving charged particle.

As seen Eq.90 uses the extended conservation equations, which assures that the radiation momentum flux is included. To show the use of these equations we will take a simple guiding example, but before that, let's define a helpful definition.

To avoid the singularity of Dirac's $\delta(r-r_0)$ function in the point r_0, we define a generalized function:

$$\Delta(\mathbf{r},\mathbf{r_0}) = \begin{cases} 0 & r > r_0 \\ \frac{1}{\mathrm{V}_0} & r \leqslant r_0 \end{cases} \tag{91}$$

The volume V_0 can be a sphere or cube or any shape which describes the body's shape, in this guiding example we use a sphere. The $\Delta(\mathbf{r},\mathbf{r_0})$ function is a distribution function without singularity, it becomes $\delta(r-r_0)$ when $r_0 \to 0$. This function 'filters out' (turns into null) any function defined in space, except for the values in the volume V_0. The impact of this distribution function is significant under integration:

$$\int f(\mathbf{x})\mathbf{\Delta}(\mathbf{x}-\mathbf{x_0},\mathbf{r_0})\mathbf{d^3x} = \frac{\mathbf{1}}{\mathbf{V_0(r_0)}}\int_{\mathbf{V_0}} \mathbf{f(x)d^3x}$$

Which is just the average value of $f(\mathbf{x})$ over the volume V_0. Abraham - Lorentz used a general external electric field, which depends on position and time. Since we want to focus on the physics of the radiation problem, and concentrate on the mechanical-electrodynamics coupling mechanism, we will take a simple external electric field.

The guiding example - In this case we take two charged bodies, one very massive compared to the other, therefore we can consider it static, we call it the static particle and the lighter one we will call the dynamic/moving particle. In the center of mass of the static body, we place the center of the reference frame, which we will use to describe the bodies and the EM field's dynamics. For simplicity, we take $|q_{static}| = |q_{dynamic}|$ or $|q_s| = |q_d|$.

Now we can start to insert the guiding example into Eq.90. We start with the fluid/mechanical tensor $T^{\mu\nu}_{fluids}$, we need to deal only with the moving particle, which becomes:

$$\begin{aligned} T^{\mu\nu}_{fluids}(\mathbf{r},\mathbf{t}) &= \rho_{\mathbf{m}}(\mathbf{r},\mathbf{r_d},\mathbf{t_r})\mathbf{v}^\mu(\mathbf{t_r})\mathbf{v}^\nu(\mathbf{t_r}) \\ &= m_{total}\Delta(\mathbf{r}-\mathbf{r_d}(\mathbf{t_r}))\mathbf{v}^\mu(\mathbf{t_r})\mathbf{v}^\nu(\mathbf{t_r}) \end{aligned} \tag{92}$$

Here, $\mathbf{r_d}(\mathbf{t_r})$ is the path of the moving particle at the retarded (past) time t_r, it is a function of the present time t by: $t_r = t - \frac{|r-r_d(t_r)|}{c}$. It is important to notice that it is only possible to measure the time t of the clock in the static frame, the point's coordinate $\mathbf{r}$ in the measuring static frame and the coordinate of the dynamic (moving) particle $\mathbf{r_d}$, these coordinates are measurable by light reflection, as Einstein's method requires. The retarded time t_r can never be measured directly, it is extracted from the function which defines it $t_r = t - \frac{|r-r_d(t_r)|}{c}$. Eq.92 describes the mechanical part of the guiding example.

Now we need to find the correct EM energy-momentum tensor $\tilde{\mathrm{T}}^{\mu\nu}_{EM}$ of this example. Since the measuring frame was chosen to be the static/massive particle frame, the massive particle's EM field is simply the electric Coulomb field $\mathbf{E_{static}}(\mathbf{r}) = \frac{\mathbf{q}}{\mathbf{r^2}}\hat{\mathbf{r}}$. To insert the moving particle's EM fields in Eq.90 correctly we need a careful procedure, since the moving particle's electric field is a Coulomb field only in its own rest frame and we need to express this EM field in the static frame. Lorentz transformation is not good enough, since accelerations are involved, the transformation we need is the retarded potentials of the moving particle. The retarded potentials are the solution of Maxwell's equations, that gives the four-potential of a moving charged particle in a rest frame relative to the moving charge. Maxwell's equations for the four-potential with Lorenz gauge $\partial_\mu \mathrm{A}^\mu = 0$ are:

$$\partial_\nu \partial^\nu \mathrm{A}^\mu = \Box \mathrm{A}^\mu = \mathrm{J}^\mu \tag{93}$$

When the four current are given, the solution of Eq.93 can be written as:

$$\mathrm{A}^\mu(\mathbf{r},\mathbf{t}) = \frac{\mathbf{1}}{\mathbf{c}}\int \frac{\mathrm{J}^\mu(\mathbf{r'},\mathbf{t'})}{|\mathbf{r}-\mathbf{r'}|}\mathbf{\Delta}(\mathbf{t'}-\mathbf{t'_r})\mathbf{\Delta}(\mathbf{r}-\mathbf{r_d}(\mathbf{t'}))\mathbf{d^3x'dt'} \tag{94}$$

For a small spherical charged body, the four-current is: $\mathrm{J}^\mu(\mathbf{r'},\mathbf{t'}) = \mathbf{q}\mathrm{v}^\mu\Delta(\mathbf{r}-\mathbf{r_d}(\mathbf{t'}))$ after integration on d^3x' Eq.94 becomes

$$\mathrm{A}^\mu(\mathbf{r},\mathbf{t}) = \frac{\mathbf{1}}{\mathbf{c}}\int \frac{\mathbf{q}\mathrm{v}^\mu(\mathrm{t'})}{|\mathbf{r}-\mathbf{r'}(\mathbf{t'})|}\mathbf{\Delta}(\mathbf{t'}-\mathbf{t'_r})\mathbf{dt'} \tag{95}$$

taking the integral on dt' we get the retarded potential of a moving spherical charged particle.

$$\varphi(\mathbf{r},\mathbf{t}) = \frac{\mathbf{q}}{(\mathbf{1}-\mathbf{n}\cdot\boldsymbol{\beta}(\mathbf{t_r}))|\mathbf{r}-\mathbf{r_d}(\mathbf{t_r})|} \tag{96}$$

and

$$\mathbf{A}(\mathbf{r},\mathbf{t}) = \frac{q\boldsymbol{\beta}}{(\mathbf{1}-\mathbf{n}(\mathbf{r},\mathbf{t_r})\cdot\boldsymbol{\beta}(\mathbf{t_r}))|\mathbf{r}-\mathbf{r_d}(\mathbf{t_r})|} \tag{97}$$

Where $\mathbf{n}$ is the unit vector between the moving particle center of mass and the measuring device at a point $\mathbf{r}$: $\mathbf{n}(\mathbf{r},\mathbf{t_r}) = \frac{\mathbf{r}-\mathbf{r_d}(\mathbf{t_r})}{|\mathbf{r}-\mathbf{r_d}(\mathbf{t_r})|}$ and the moving particle's velocity divided by c is $\beta = \frac{\mathbf{v}(\mathbf{t_r})}{\mathbf{c}}$.

We can get the form of the retarded electric and magnetic fields by their definition: $\mathbf{H} = \nabla\times\mathbf{A}$ and $\mathbf{E} = \nabla\varphi - \frac{\partial\mathbf{A}}{\mathbf{c}\partial\mathbf{t}}$, the derivatives process has to be taken carefully remembering that the retarded time $t_r = t - \frac{|r-r_d(t_r)}{c}|$, is also a function of coordinates. The retarded fields $\mathbf{E_r}$ and $\mathbf{H_r}$ in the non relativistic case ($\gamma \approx 1$) are:

$$\mathbf{E_r}(\mathbf{r},t) = \left(\frac{q(\mathbf{n}-\boldsymbol{\beta})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r}-\mathbf{r}_d|^2} + \frac{q\mathbf{n}\times\Big((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\Big)}{c(1-\mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r}-\mathbf{r}_d|}\right)_{t_r} \tag{98}$$

$$\mathbf{H_r}(\mathbf{r},t) = \left(\frac{q\boldsymbol{\beta}\times\mathbf{n}}{c(1-\mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r}-\mathbf{r}_d|^2} + \frac{q\mathbf{n}\times\Big(\mathbf{n}\times\Big((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\Big)\Big)}{c^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3|\mathbf{r}-\mathbf{r}_d|}\right)_{t_r} \tag{99}$$

The unknown quantity is the specific path of the moving particle $\mathbf{r_d}(\mathbf{t_r})$. If the path is known, its derivatives $\boldsymbol{\beta}(t_r) = \frac{d\mathbf{r_d}(\mathbf{t_r})}{cdt_r} = [\frac{d\mathbf{r_d}(\mathbf{t})}{cdt}]_{t_r}$ and $\dot{\boldsymbol{\beta}}(t_r) = [\frac{d\boldsymbol{\beta}(t)}{dt}]_{t_r}$ are also known.

We have to notice, the total EM field that will be measured by a field detector in the static frame, is the combination of two fields, the static particle's and the moving particle's field: $\mathbf{E} = \mathbf{E_{static}} + \mathbf{E_r}$ and $\mathbf{H} = \mathbf{H_{static}} + \mathbf{H_r}$. Now we have to deal with another problem that a charged particle possesses: its core is discrete, meaning it can be described by path of a point like particle, which is defined by discrete quantities like the particle mass and charge. On the other hand it possesses an EM field which moves with its center of mass but can change for an external observer as retarded $\mathbf{E}$ and $\mathbf{H}$ suggest. The mechanical and EM parts will be merged using the Δ function and integration. To explain the merging procedure we start with example of the mechanical and EM mass density:

$$\rho(t,\mathbf{r}) = \frac{\mathbf{m}}{\mathbf{V_0}}\boldsymbol{\Delta}(\mathbf{r}-\mathbf{r_d}) + \tilde{\rho}(t,\mathbf{r}) \tag{100}$$

We can check that for $\mathbf{r}$ inside the massive core volume V_0 the density $\rho(t,\mathbf{r}) \approx \frac{\mathbf{m}}{\mathbf{V_0}}$ since $|\tilde{\rho}(t,\mathbf{r})| << \frac{\mathbf{m}}{\mathbf{V_0}}$. For $|\mathbf{r}| > \mathbf{R_{core}}$ the density is only the electromagnetic $\rho(t,\mathbf{r}) = \tilde{\rho}(t,\mathbf{r})$ which is small compared to the core mass density. Taking the integral of Eq.100, over a large enough volume V which includes the dynamic particle in all the relevant path, we get:

$$\int \rho(t,\mathbf{r})\mathbf{d^3r} = \mathbf{m} + \int \tilde{\rho}(\mathbf{t},\mathbf{r})\mathbf{d^3r}$$

The integral gives the particle's core mass plus its EM fields' mass. In most cases the EM mass can be neglected but in our guiding example we should keep it to explore the time development of the EM fields and their inertia.

Now we can investigate the continuity equation, which is the first equation out of four in Eq.90:

$$\frac{m\partial(\Delta(\mathbf{r}-\mathbf{r_d}(\mathbf{t_r}),\mathbf{r_0})}{c\partial t} + m\nabla\cdot[(\Delta(\mathbf{r}-\mathbf{r_d}(\mathbf{t_r}),\mathbf{r_0})\mathbf{v}(\mathbf{t_r})] + \frac{\partial(\tilde{\rho})_{t_r}}{c\partial t} + \nabla\cdot(\tilde{\rho}\tilde{\mathbf{u}})_{t_r} = 0 \tag{101}$$

Using the definition of the Δ function above, the term $\frac{\partial\Delta(\mathbf{r},\mathbf{r_0})}{\partial r_i} = 0$ at any point in space except on the particle's surface $r = r_0$ where it diverges to infinity as $\frac{1}{r^2}$. Its partial time derivative , by the chain rule, $\frac{\partial(\Delta(\mathbf{r},\mathbf{r_0})}{\partial r_i}\frac{\partial r_i}{\partial t} = 0$ in any point except on the particle's surface $r = r_0$ where it diverges to infinity as $\frac{1}{r^2}v^i(t)$.

To avoid these singularities, we will integrate over the volume. The integration removes the dependency on $\mathbf{r}$ and changes Eq.90 from partial deferential equations to regular deferential equations depending only on t. In our guiding example, the external field is not time dependent, only the particle path $\mathbf{r_d}(\mathbf{t})$ is time dependent. For a start, we integrate both sides of Eq.100:

$$\begin{aligned}\frac{m\partial}{c\partial t}\int_V \Delta(\mathbf{r'}-\mathbf{r_d}(\mathbf{t_r}),\mathbf{r_0})\mathbf{d^3r'} + \\ m\nabla\cdot\int_V[(\Delta(\mathbf{r'}-\mathbf{r_d}(\mathbf{t_r}),\mathbf{r_0})\mathbf{v}(\mathbf{t_r})]\mathbf{d^3r'} + \\ \frac{\partial}{c\partial t}\int_V \tilde{\rho}d^3\mathbf{r'} + \int_{\mathbf{V}}\nabla\cdot(\tilde{\rho}\tilde{\mathbf{u}})\,\mathbf{d^3r'} = \mathbf{0}\end{aligned} \tag{102}$$

The first and the second terms in Eq.102 are null, since $\int_V \Delta(\mathbf{r'}-\mathbf{r_d}(\mathbf{t_r}),\mathbf{r_0})\mathbf{d^3r'} = \mathbf{1}$ and Eq.102 becomes:

$$\frac{\partial}{c\partial t}\int_V \tilde{\rho}d^3\mathbf{r'} + \int_{\mathbf{V}}\nabla\cdot(\tilde{\rho}\tilde{\mathbf{u}})\,\mathbf{d^3r'} = \mathbf{0} \tag{103}$$

This is the continuity equation for the EM energy/mass. This equation is the constrain which demands that the change of energy/mass within the volume is equal to the energy/mass flowing out, and this is the EM energy that escapes the volume.

The other three equations in Eq.90 in vector form are:

$$\frac{\partial[(m\Delta(\mathbf{r}-\mathbf{r_d}(t_\mathbf{r}),\mathbf{r_0})+\tilde{\rho}(\mathbf{t},\mathbf{x}))\mathbf{v}]_{\mathbf{t_r}}}{\partial t} + c\nabla\cdot[(m\Delta(\mathbf{r}-\mathbf{r_d}(t_r),\mathbf{r_0})] + \tilde{\rho}(t,\mathbf{x}))\mathbf{v}(\mathbf{t})]_{\mathbf{t_r}} + \partial_\mathrm{t}\,[\rho\tilde{\mathbf{u}}]_{\mathbf{t_r}} + \partial_\mathbf{i}[\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^\mathbf{i}]_{\mathbf{t_r}} =$$

$$\frac{1}{4\pi c}[\mathbf{E}\nabla\cdot\mathbf{E}+\mathbf{H}\nabla\cdot\mathbf{H}-\mathbf{E}\times(\nabla\times\mathbf{E})-\mathbf{H}\times(\nabla\times\mathbf{H})]_{\mathbf{t_r}} \tag{104}$$

We are interested in the particle's center of mass motion, therefore, we integrate over a large enough volume, so the moving particle is always inside. Integrating the other three equations of Eq.104 gives:

$$\begin{gathered}\frac{\partial}{\partial t}\int_V m\Delta(\mathbf{r'}-\mathbf{r_d}(\mathbf{t_r}),\mathbf{r_0})\mathbf{v}(\mathbf{t_r})\mathbf{d^3r'}+\\ \frac{\partial}{\partial t}\int_V\tilde{\rho}(t_r,\mathbf{r'})\mathbf{v}(\mathbf{t_r})\mathbf{d^3r'}+\\ \int_V\partial_i[(m\Delta(\mathbf{r'}-\mathbf{r_d}(\mathbf{t_r}),\mathbf{r_0})+\tilde{\rho}(\mathbf{t_r},\mathbf{r'})]\mathbf{v}(\mathbf{t_r})\mathbf{v^i}(\mathbf{t_r})\mathbf{d^3r'}+\\ \frac{\partial}{\partial t}\int_V\tilde{\rho}(t_r,r')\tilde{\mathbf{u}}(t_r,r')d^3\mathbf{r'}+\\ \int_V\partial_i[\tilde{\rho}(t_r,r')\tilde{\mathbf{u}}(\mathbf{t_r},\mathbf{r'})\tilde{\mathbf{u}}^{\mathbf{i}}(\mathbf{t_r},\mathbf{r'})]\mathbf{d^3r'}=\\ \int_V\left[\frac{\mathbf{E}\nabla\cdot\mathbf{E}}{4\pi c}+\frac{\mathbf{H}\nabla\cdot\mathbf{H}}{4\pi c}-\frac{\mathbf{E}\times(\nabla\times\mathbf{E})}{4\pi c}-\frac{\mathbf{H}\times(\nabla\times\mathbf{H})}{4\pi c}\right]_{(t_r,\mathbf{r'})}d^3\mathbf{r'}\end{gathered} \tag{105}$$

Since retarded time $t_r=t-\frac{|r-r_d|}{c}$ depends on $\mathbf{r}$, these integrals are difficult to evaluate in their present form, so we will rewrite them by replacing t_r with t' and integrating over the delta distribution $\delta(t'-t_r)$ therefore for any vector function of $\mathbf{f}(\mathbf{t_r},\mathbf{r})$:

$$\int_V\mathbf{f}(\mathbf{t_r},\mathbf{r'})\mathbf{d^3r'}=\int\int_{\mathbf{V}}\mathbf{f}(\mathbf{t_r},\mathbf{r'})\delta(\mathbf{t'}-\mathbf{t_r})\mathbf{dt'd^3r'}$$

To evaluate this integral we need the identity:

$$\delta(f(t))=\Sigma_i\frac{\delta(t-t_i)}{|f'(t_i)|}$$

where each t_i is a root of $f(t)$ i.e. $f(t_i)=0$. In our example there is only one root for a given t_r, therefore:

$$\delta(t'-t'_r)=\frac{\delta(t'-t'_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}$$

Since r' and t' are not coupled, we can exchange the integration order, first on r' and than over t', :

$$\int_V\mathbf{g}(\mathbf{t_r},\mathbf{r'})\mathbf{d^3r'}=\int\frac{\delta(\mathbf{t'}-\mathbf{t_r})}{\mathbf{1}-\boldsymbol{\beta}_\mathbf{d}\cdot\mathbf{n}}\mathbf{dt'}\int_{\mathbf{V}}\mathbf{g}(\mathbf{t'},\mathbf{r'})\mathbf{d^3r'} \tag{106}$$

Using Eq.106 on the first left term in Eq.105 gives:

$$\frac{\partial}{\partial t}\left[\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\int_V m\Delta(\mathbf{r'}-\mathbf{r_d}(\mathbf{t'}),\mathbf{r_0})\mathbf{v}(\mathbf{t'})\mathbf{d^3r}\right]$$

$$=\frac{m\dot{\mathbf{v}}(\mathbf{t_r})}{(1-\boldsymbol{\beta}_d\cdot\mathbf{n})^{\mathbf{2}}}$$

The second term in Eq.105 is made of a Newtonian velocity and the EM mass density inside the integrated volume: $\frac{\partial}{\partial t}\left[\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\int_V\tilde{\rho}(t',\mathbf{r}))\mathbf{v}(\mathbf{t'})\mathbf{d^3r}\right]$, and can be written as:

$$\frac{\partial}{\partial t}\left[\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}v(t')dt'\int_V\tilde{\rho}(t',\mathbf{r}))\mathbf{d^3r}\right]$$

or

$$\frac{\partial}{\partial t}\left[\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}v(t')\tilde{\rho}(t')dt'\right]$$

After the time integration we get:

$$\frac{\partial}{\partial t}\left[\frac{v(t_r)\tilde{\rho}(t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}\right]=\frac{\dot{v}(t_r)\tilde{\rho}(t_r)+v(t_r)\dot{\tilde{\rho}}(t_r)}{(1-\boldsymbol{\beta}_d\cdot\mathbf{n})^{\mathbf{2}}}$$

The term $\dot{v}(t_r)\tilde{\rho}(t_r)$ is the Newtonian inertia force emerging from EM field's mass $U_{EM}/c^2\equiv\delta m_{EM}$ and the mechanical acceleration $\dot{\mathbf{v}}(\mathbf{t})$ of its source. For this classical example $\delta m_{EM}<<m$, therefore for our investigation the term δm_{EM} can be neglected.

The term $\dot{\tilde{\rho}}(t_r)$ is the change rate of the EM mass density of the particle's fields, which in this classical case is very small compared to the other terms and therefore can be neglected.

The third term in Eq.105 is converted to a surface integral:

$$\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\mathbf{v}(\mathbf{t'})\mathbf{v^i}(\mathbf{t'})\int_{\mathbf{S}}[(\mathbf{m}\boldsymbol{\Delta}(\mathbf{r}-\mathbf{r_d}(\mathbf{t'}))+\tilde{\rho}(\mathbf{t'},\mathbf{x}))]\mathbf{d^2}\sigma_{\mathbf{i}}$$

on a large enough surface the mass density is null and this third term is null.

The fourth term:

$$\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\frac{\partial}{\partial t}\int_V\tilde{\rho}(t_r,r')\tilde{\mathbf{u}}(t_r,r')d^3\mathbf{r'}$$

can not be neglected since it represents the EM-momentum change per unit time, which is part of the EM field impact on the particle's acceleration.

The term:

$$\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\int_V\partial_i[\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^{\mathbf{i}}(\mathbf{t'},\mathbf{r'})]\mathbf{d^3r'}$$

which is the EM fields' momentum flux, can also be written as a surface integral:

$$\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\int_S[\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^{\mathbf{i}}(\mathbf{t'},\mathbf{r'})]\mathbf{d^2}\sigma_{\mathbf{i}}$$

This integral represents the EM flux which leaves the surface and does not return, this is the definition of radiation.

Summarising all of the above Eq.105 becomes:

$$\begin{gathered}\frac{m\dot{\mathbf{v}}(\mathbf{t_r})}{(1-\boldsymbol{\beta}_d\cdot\mathbf{n})^{\mathbf{2}}}+\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\frac{\partial}{\partial t}\int_V\tilde{\rho}(t_r,r')\tilde{\mathbf{u}}(t_r,r')d^3\mathbf{r'}\\ +\int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\int_V\partial_i[\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^{\mathbf{i}}(\mathbf{t'},\mathbf{r'})]\mathbf{d^3r'}=\\ \int\frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\int_V[\frac{\mathbf{E}\nabla\cdot\mathbf{E}}{4\pi}+\frac{\mathbf{H}\nabla\cdot\mathbf{H}}{4\pi}-\frac{\mathbf{E}\times(\nabla\times\mathbf{E})}{4\pi}-\frac{\mathbf{H}\times(\nabla\times\mathbf{H})}{4\pi}]_{(t',\mathbf{r'})}d^3\mathbf{r'}\end{gathered} \tag{107}$$

Eq.107 holds only the significant terms giving the clearest possible picture, without loosing the important aspects of radiation reaction. The left hand-side of this equation represents the mechanical and EM inertial forces, these EM inertial forces originate from the divergence of the EM stresses. As mentioned above, the EM fields in any point are a combination of the stationary particle's field and the moving particle's retarded field, we symbolize the fields $\mathbf{E} \equiv \mathbf{E_{total}} = \mathbf{E_{static}} + \mathbf{E_{retarded}}$ and since the static particle does not have a magnetic field $\mathbf{H} \equiv \mathbf{H_{total}} = \mathbf{H_{static}} + \mathbf{H_{retarded}} = \mathbf{H_{retarded}}$, the static particle's field is a simple Coulomb field $\mathbf{E_{static}}(\mathbf{r}) = \frac{\mathbf{q}}{\mathbf{r^2}}\hat{\mathbf{r}}$, while the moving particle's fields $\mathbf{E_{retarded}}$ and $\mathbf{H_{retarded}}$ as explained are the Coulomb field of the accelerated particle as measured in the static frame. As a reminder, Eq.107 is the motion equation that has to be solved in order to find the trajectory $\mathbf{r_d}(\mathbf{t})$ as function of the time t in the static frame.

Since, $\dot{\tilde{p}} = \frac{\mathbf{E_r}\times\dot{\mathbf{H}}_\mathbf{r}}{4\pi c} - \frac{\mathbf{H_r}\times\dot{\mathbf{E}}_\mathbf{r}}{4\pi c}$ and using Maxwell's equation: $\nabla \times \mathbf{E_r} = \frac{\mathbf{1}}{\mathbf{c}}\frac{\partial \mathbf{H_r}}{\partial \mathbf{t}}$, insert it into the term $\frac{\mathbf{E}\times(\nabla\times\mathbf{E})}{4\pi}$ which becomes $\frac{\mathbf{E_r}\times\dot{\mathbf{H}}_\mathbf{r}}{4\pi c}$. This Maxwell's equation $\nabla \times \mathbf{H_r} = \frac{\mathbf{1}}{\mathbf{c}}\frac{\partial \mathbf{E_r}}{\partial \mathbf{t}} + \mathbf{4\pi j}$ is inserted into $\frac{\mathbf{H}\times(\nabla\times\mathbf{H})}{4\pi}$, which becomes $\frac{\mathbf{H}\times(\frac{\mathbf{1}}{\mathbf{c}}\frac{\partial \mathbf{E_r}}{\partial \mathbf{t}}+\mathbf{4\pi j})}{4\pi}$. We get similar terms on both sides which cancel each other, therefore Eq.107 becomes:

$$\begin{aligned}\int \frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'[\frac{m\dot{\mathbf{v}}(\mathbf{t}')}{(1-\boldsymbol{\beta}_d\cdot\mathbf{n})}] \\ +\int \frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\int_V \partial_i[\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^\mathbf{i}(\mathbf{t}',\mathbf{r}')]\mathbf{d^3r'} \\ = \int \frac{\delta(t'-t_r)}{1-\boldsymbol{\beta}_d\cdot\mathbf{n}}dt'\int_V (\mathbf{E_s}\nabla\cdot\mathbf{E_r} - \mathbf{H_r}\times\mathbf{j})\mathbf{d^3r'}\end{aligned} \tag{108}$$

The last simplification is $\int_V(\mathbf{H_r}\times\mathbf{j})\mathbf{d^3r} = \mathbf{0}$, since it represents the overall forces of the moving particle magnetic fields on itself. If it was not null, it would mean that the particle can induce force on itself that will accelerate it even without the existence of an external force, therefore, it must be null. Taking the time integral, changes $t' \to t_r$, therefore Eq.108 becomes:

$$\frac{m\dot{\mathbf{v}}(\mathbf{t_r})}{(1-\boldsymbol{\beta}_d\cdot\mathbf{n})^\mathbf{2}} + \int_V \frac{\partial_i[\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^\mathbf{i}(\mathbf{t_r},\mathbf{r}')]}{(1-\boldsymbol{\beta}_d\cdot\mathbf{n})}d^3\mathbf{r'} = \int_\mathbf{V}\frac{\mathbf{E_s}\nabla\cdot\mathbf{E_r}(\mathbf{t_r},\mathbf{r}')}{(\mathbf{1}-\boldsymbol{\beta}_\mathbf{d}\cdot\mathbf{n})}\mathbf{d^3r'} \tag{109}$$

The first term in Eq.109 is the Newtonian core mass inertial force with a small correction, which comes from the retarded time derivative $\frac{dt_r}{dt} = \frac{1}{(1-\boldsymbol{\beta}_d\cdot\mathbf{n})}$, therefore:

$$\begin{aligned}\frac{d^2\mathbf{r}(\mathbf{t_r})}{dt^2} = \frac{d}{dt}(\frac{d\mathbf{r}(\mathbf{t_r})}{dt_r}\frac{dt_r}{dt}) = \frac{d^2\mathbf{r}(\mathbf{t_r})}{dt^2{}_r}\frac{1}{(1-\boldsymbol{\beta}_d\cdot\mathbf{n})^\mathbf{2}} + \frac{d\mathbf{r}(\mathbf{t_r})}{dt_r}\frac{d^2t_r}{dt^2} \\ = \frac{\dot{\mathbf{v}}}{(1-\boldsymbol{\beta}_d\cdot\mathbf{n})^\mathbf{2}} + \mathbf{v}\frac{\mathbf{d^2t_r}}{\mathbf{dt^2}}\end{aligned}$$

The term $\frac{d^2t_r}{dt^2}$ is proportional to $\frac{1}{c}$ therefore negligible. Usually, retarded time appears in electrodynamics radiation cases, but the retarded time is the correction for any physical quantity of a moving particle in other frames of reference which take into account also acceleration.

Taking the partial derivatives in both sides of Eq.110 is long and cumbersome, since the retarded time is function of the coordinates.

The divergence of $\mathbf{E_{retarded}}$ taking $(1-\mathbf{n}\cdot\boldsymbol{\beta}) \approx 1$ is:

$$\begin{aligned}\nabla\cdot\mathbf{E_r} = \mathbf{q\Delta}(\mathbf{r}-\mathbf{r}(\mathbf{t_r}))(\mathbf{1}-\mathbf{n}\cdot\boldsymbol{\beta})^{-\mathbf{3}} + \\ \frac{2q[\ddot{\boldsymbol{\beta}}\cdot\mathbf{n}+(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^\mathbf{2}]}{c^2|\mathbf{r}-\mathbf{r}_d|} + \frac{2q(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})-3q(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})(\mathbf{n}\cdot\boldsymbol{\beta})}{c|\mathbf{r}-\mathbf{r}_d|^2} \\ + \frac{3q[(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})(\mathbf{n}\cdot\boldsymbol{\beta})-(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})-\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}^\mathbf{2}+(\mathbf{n}\cdot\boldsymbol{\beta})^\mathbf{2}+\boldsymbol{\beta}^\mathbf{2})]-\mathbf{8q}(\mathbf{n}\cdot\boldsymbol{\beta})}{c|\mathbf{r}-\mathbf{r}_d|^3}\end{aligned} \tag{110}$$

When $\beta \to 0$, which means $t_r \to t$, the only term left in Eq.109 is the Delta function $q\Delta(\mathbf{r}-\mathbf{r}(\mathbf{t}))$ describes an homogeneous sphere charge distribution, as we excepted. When the sphere radius goes to zero it becomes a point charge distribution $q\delta(\mathbf{r}-\mathbf{r}(\mathbf{t}))$, it is the only force term used by Abraham - Lorentz model. These terms describe the moving particle's charge distribution as will be measured in the rest frame, taking in account the relative distance, velocity, acceleration and its time derivative.

The charge distribution $q\Delta(\mathbf{x}-\mathbf{x_s}(\mathbf{t}))$ will be named bare charge density, all the terms in Eq.109 will be named the 'extra' charge density or non-bare charge density, symbolised by: $\delta\rho_q$. Therefore, we can rewrite: $\nabla_r\cdot\mathbf{E_{retarded}} = \mathbf{q\Delta}(\mathbf{x}-\mathbf{x_s}(\mathbf{t})) + \delta\rho_\mathbf{q}$. Now Eq.109 is rewritten as:

$$\begin{aligned}m\dot{\mathbf{v}}(\mathbf{t_r}) + \int_\mathbf{V}\partial_\mathbf{i}[\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^\mathbf{i}(\mathbf{t_r},\mathbf{r}')]\mathbf{d^3r'} \\ = q\mathbf{E_s} + \int_\mathbf{V}\mathbf{E_s}\delta\rho_\mathbf{q}(\mathbf{t_r},\mathbf{r}')\mathbf{d^3r'}\end{aligned} \tag{111}$$

The additional forces that the moving particle experiences come from the integral $\int_V \mathbf{E_s}\delta\rho_\mathbf{q}(\mathbf{t_r},\mathbf{r}')\mathbf{r'^2dr'd\Omega}$, where in this case $\mathbf{E_s} = \frac{\mathbf{q_s\hat{r}}}{\mathbf{r^2}}$. Therefore the extra force is:

$$\int_V q_s\delta\rho_q(t_r,\mathbf{r}')\mathbf{dr'd\Omega}$$

Now we analyze the left side of Eq.111, the terms $\partial_i(\tilde{\mathbf{p}}\tilde{\mathbf{u}}^\mathbf{i}) = \partial_\mathbf{i}(\frac{\tilde{\mathbf{p}}\tilde{\mathbf{p}}^\mathbf{i}}{\tilde{\rho}})$ influence becomes clearer in cylindrical coordinates, as already written in equations Eq.47-49. In these equations we see the EM inertial forces, like the centrifugal EM forces and Coriolis EM forces. The exact form of these terms, for our guiding example, is found by installing the total EM momentum: $\tilde{\mathbf{p}} = \frac{1}{4\pi c}\left[(\mathbf{E_{static}}+\mathbf{E_{moving}})\times(\mathbf{H_{static}}+\mathbf{H_{moving}})\right]$ and $\tilde{\rho} = \frac{(\mathbf{E_{static}}+\mathbf{E_{moving}})^\mathbf{2}+(\mathbf{H_{static}}+\mathbf{H_{moving}})^\mathbf{2}}{8\pi c^2}$ and taking the ∂_i derivatives. This procedure in very long, therefore, we only investigate the derivatives terms which describe the radiation, these term must decay as $\frac{1}{|r-r_s|^2}$.

The EM momentum density in our case is:

$$\begin{aligned}\tilde{\mathbf{p}} = \frac{1}{4\pi c}\left[(\mathbf{E_{static}}+\mathbf{E_r})\times(\mathbf{H_{static}}+\mathbf{H_r})\right] = \\ \frac{1}{4\pi c}\left[\mathbf{E_{static}}\times\mathbf{H_r}+\mathbf{E_r}\times\mathbf{H_r}\right]\end{aligned}$$

and the EM energy density:

$$\tilde{\rho} = \frac{(\mathbf{E_{static}}+\mathbf{E_r})^\mathbf{2}+(\mathbf{H_r})^\mathbf{2}}{8\pi c^2}$$

Since $\mathbf{H_{static}} = \mathbf{0}$ and the retarded fields $\mathbf{H_r} = \mathbf{E_r} \times \frac{\mathbf{n}}{\mathbf{c}}$ and $\mathbf{E_{static}} \propto \mathbf{r_d^{-2}}$, far from the moving particle the momentum is:

$$\tilde{\mathbf{p}} = \frac{1}{4\pi c}\mathbf{E_r} \times (\mathbf{E_r} \times \frac{\mathbf{n}}{\mathbf{c}}) = \frac{\mathbf{1}}{\mathbf{4\pi c^2}}[(\mathbf{E_r}\cdot\mathbf{n})\mathbf{E_r} - (\mathbf{E_r}\cdot\mathbf{E_r})\mathbf{n}] = \frac{q^2}{4\pi c^2}|\mathbf{E_r}|^{\mathbf{2}}\mathbf{cos}(\eta)\hat{\mathbf{E}}_{\mathbf{r}} - \frac{\mathbf{q^2}}{\mathbf{4\pi c}}\left(\frac{\mathbf{r}\cdot(\mathbf{n}-\boldsymbol{\beta})}{|\mathbf{r}-\mathbf{r_d}|^{\mathbf{2}}|\mathbf{r}|^{\mathbf{3}}} + \frac{(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})(\mathbf{r}\cdot\mathbf{n}) - (\mathbf{r}\cdot\dot{\boldsymbol{\beta}}) - (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})(\mathbf{r}\cdot\boldsymbol{\beta}) + (\mathbf{n}\cdot\boldsymbol{\beta})(\mathbf{r}\cdot\dot{\boldsymbol{\beta}})}{c|\mathbf{r}-\mathbf{r}_d||\mathbf{r}|^3}\right)\mathbf{n}$$

Where η is the angel between $\mathbf{n}$ and $\mathbf{E_r}$, the last equation can be written as:

$$\tilde{\mathbf{p}} = \frac{1}{4\pi c^2}[(\mathbf{E_r}\cdot\mathbf{n})\hat{\mathbf{E_r}} - (\mathbf{E_r}\cdot\mathbf{E_r})\hat{\mathbf{n}}] = \frac{\mathbf{1}}{\mathbf{4\pi c^2}}\mathbf{L}[\mathbf{cos}(\eta)\hat{\mathbf{E}}_{\mathbf{r}} - \hat{\mathbf{n}}]$$

Where:

$$L \equiv -\frac{1-2\mathbf{n}\cdot\boldsymbol{\beta}+\beta^2}{|\mathbf{r}-\mathbf{r}_d|^4} - 2\frac{(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}\cdot\mathbf{n} - \dot{\boldsymbol{\beta}}\cdot\boldsymbol{\beta} - (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}^{\mathbf{2}} + (\mathbf{n}\cdot\boldsymbol{\beta})\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}}{c|\mathbf{r}-\mathbf{r}_d|^3} + \frac{\dot{\boldsymbol{\beta}}^2 - (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2 + (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) + (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2\boldsymbol{\beta}^2 - (\mathbf{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}}^2 - (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})(\mathbf{n}\cdot\boldsymbol{\beta})(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})}{c^2|\mathbf{r}-\mathbf{r}_d|^2}$$

Using the same arguments above and since the term $\mathrm{H_r^2} = (\frac{\mathrm{n}}{\mathrm{c}} \times \mathrm{H_r})^2 \approx \frac{1}{\mathrm{c}^2}\mathrm{E_r^2}$ this means $\mathrm{H_r^2} << \mathrm{E_r^2}$ and can be neglected, therefore, the EM density becomes:

$$\tilde{\rho} \approx [\frac{\mathrm{E_r^2}}{8\pi c^2}] = \frac{L}{8\pi c^2}$$

Now we can estimate the momentum flux terms:

$$\tilde{\mathbf{p}}^{\mathbf{i}}\tilde{\mathbf{p}}^{\mathbf{j}}/\tilde{\rho} \approx \frac{\mathbf{1}}{\mathbf{2\pi c^2}}L[\mathbf{cos}(\eta)\hat{\mathbf{E}}_{\mathbf{r}}^{\mathbf{i}} - \hat{\mathbf{n}}^{\mathbf{i}}][\mathbf{cos}(\eta)\hat{\mathbf{E}}_{\mathbf{r}}^{\mathbf{j}} - \hat{\mathbf{n}}^{\mathbf{j}}] \quad (112)$$

using Gauss surface integral of the second term in Eq.111: $\int_V \partial_i[\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^{\mathbf{i}}(\mathbf{t_r},\mathbf{r}')]\mathbf{d^3r'}$, gives:

$$f^i = \int_S \frac{1}{2\pi c^2}L[cos(\eta)\hat{\mathbf{E}}_r^i - \hat{\mathbf{n}}^i][cos(\eta)\hat{\mathbf{E}}_r^j - \hat{\mathbf{n}}^j]d^2\sigma_j \quad (113)$$

This is the total EM momentum flux on the surface, which creates a force f^i on the particle . These terms are missing in Abraham -Lorentz motion equation.

In order to investigate Eq.113 we take a simple example of a moving particle in a circle with constant velocity, which gives a constant acceleration on the direction of the vector between the static source particle and the moving particle, which is $\hat{\mathbf{r}}_d$. Since $\dot{\beta} = \mathbf{a} = \; Constant$, therefore in circular motion $(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) = 0$, therefore L becomes:

$$L'(\mathbf{r},\mathbf{r_d},\boldsymbol{\beta},\dot{\boldsymbol{\beta}}) \equiv -\frac{\mathbf{1-2n}\cdot\boldsymbol{\beta}+\beta^{\mathbf{2}}}{|\mathbf{r}-\mathbf{r_s}|^{\mathbf{4}}} - \mathbf{2}\frac{(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}\cdot\mathbf{n} - (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}^{\mathbf{2}}}{\mathbf{c}|\mathbf{r}-\mathbf{r_s}|^{\mathbf{3}}} + \frac{-\dot{\boldsymbol{\beta}}^2 + (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2 - (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2\boldsymbol{\beta}^2 + (\mathbf{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}}^2}{c^2|\mathbf{r}-\mathbf{r}_s|^2}$$

The integral Eq.113, taken on a sphere with radius R is:

$$\frac{q^2}{2\pi c^2}\int_S L'[cos(\eta)\hat{\mathbf{E}}_r - \hat{\mathbf{n}}][cos(\eta)\hat{\mathbf{E}}_r\cdot\hat{\mathbf{R}} - \hat{\mathbf{n}}\cdot\hat{\mathbf{R}}]R^2 sin(\theta)d\theta d\phi \quad (114)$$

When $R >> r_d$ the $\hat{\mathbf{n}} \to \hat{\mathbf{R}}$, the only term left far from the moving particle is $\frac{-(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2\boldsymbol{\beta}^2}{c^2|\mathbf{r}-\mathbf{r}_s|^2}$. The integral of this term is proportional to $\frac{2q^2}{c^4}a^2\beta^2$. Translating this force into power by definition is $W = \mathbf{F}\cdot\mathbf{v}$, taking $\mathbf{v} = \mathbf{c}$ we get $W = \frac{2q^2}{c^4}a^2\beta^2\mathbf{n}\cdot\mathbf{c} = \frac{\mathbf{2q^2}}{\mathbf{c^3}}\mathbf{a^2}\beta^{\mathbf{2}}\mathbf{n}\cdot\hat{\mathbf{c}}$, compared to the Larmor radiation power: $\frac{2q^2}{3c^3}a^2$, we can see that they are identical except for $\beta^2/3$.

The open question arising from the contradiction between Abraham-Lorentz approach and Larmor is whether such a particle will radiate. We can conclude that a perfect circulating particle will radiate.

The forces coming from the partial derivative of $\partial_i(\tilde{\mathbf{p}}\tilde{\mathbf{u}}^{\mathbf{i}})$ are probably hard to detect directly, but in special cases where extreme EM fields are involved, these forces can become dominant.

Even this simple one dynamical particle motion equation is not simple to solve, which shows how complicated nature is when EM fields interaction and EM inertia are taken in account. We have to remember we have used few assumptions to simplify the exact motion equations Eq.90, the major one was the use of the retarded EM field. We did not take the total influence that of the extended equations on the propagation of the EM fields, only their momentum flux assuming the propagation is Maxwellian. But when extreme EM fields are involved and the wave length is less than the characteristic length of the problem, Maxwellian propagation is not good enough and more precise solution of Eq.90 should be considered.

Summery of Abraham Lorentz problem:

To solve the radiation paradox we used Eq.90 and chose the simple case of massive particle and a small moving particle to clarify the physical picture in which the particles and their fields are dynamically coupled and dependent on the acceleration as well as distance and velocity. The coupling, after some simplification, is represented by Eq.111, in this equation the retarded EM fields are a function of the particle's path. Solving Eq.111 will give us the path of the moving particle and inserting it back into the retarded field will give us the time propagation of the field.

We can say with confidence, that the term $\frac{2e^2}{3c^3}\dot{\mathbf{a}}$ in Abraham Lorentz equation does not represent correctly the physics of accelerating charge even for such a simple guiding example.

Since validation of any new theory has to be done in experiments, it might be possible to check the accuracy of these new motion equations of charged particles in labs of high energy lasers.

X. CONCLUSION

We began by defining a new EM momentum proper tensor without energy density term and without Maxwell's stresses terms. This enabled us to define a proper scalar energy-mass density and define the EM momentum flux covariant tensor. Using these new definitions enabled us to write the generalized Minkowski energy momentum tensor to include the momentum flux that exists in any continuum, mechanical or electromagnetic, but was not represented in the original Minkowski's energy momentum tensor. This new energy momentum tensor that includes the momentum flux terms, gives new energy momentum conservation equations. These four new equations together with the four equations $\partial_\mu \mathfrak{F}^{\mu\nu} = 0$, (which are a mathematical constrain assuring the EM fields are a four-rotor of a four-vector), are the extended motion equations of electromagnetic fields. We proved the equivalence between these four Maxwell's equations $\partial_\mu \mathrm{F}^{\mu\nu} = 4\pi \mathrm{J}^\nu$ and the Minikowski EM conservation equations. Using this equivalence, we found the extended four Maxwell's equations which are equivalent to the extended conservation equations. Next we found the Lagrangian of the extended equations and brief comparison to other nonlinear extensions of EM Lagrangians. To prove the validity for these new EM definitions and extended equations, we presented the solution of known paradoxes and problems in electrodynamics that have perplexed the physics community for decades. Using the new proper definitions of momentum flux and EM energy density scalar, solved the 4/3 problem, by itself. The example of a long solenoid with a radial electric field is used to emphasize the lack of representation of inertial forces like centrifugal forces acting on EM fields within the vacuum. These forces come out naturally and straightforward from the new extended equations. The last example we solved is the known Abraham Lorentz radiation problem. For clear physical picture we took a simple example of two charged particles, one small particle which is accelerated by the other massive and static particle. It is important to note that the particles and their fields are dynamically coupled and depend on the acceleration as well as distance and velocity.

The extended Maxwell's equations above, did not use any new constants and are non-linear, therefore, can predict phenomena related to field-field interaction, like light bending (or even confinement) by strong magnetic field as pulsars or Magnatars.

Further investigation of the extended equations and some surprising results will be presented in following papers.

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